Optimal Low Thrust Spacecraft Maneuvers for Station Keeping

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Abstract: This paper studies orbital maneuvers for a spacecraft that has the goal of performing station keeping in its orbit. This type of maneuver is necessary in order to compensate the effects of perturbations in the orbit of a spacecraft that needs to stay in a specific orbit to be able to perform its mission. It will use a low thrust to achieve this task. To do this, it is also important to find solutions that minimize the fuel consumption, since fuel will also be required for other tasks of the mission. To solve this problem, the present research uses a hybrid optimal control approach to numerically approach the problem, where the question of the accuracy of the satisfaction of the constraints is considered. The spacecraft is assumed to have its motion controlled by the gravity field of the Earth and the forces delivered by the thrusts. These thrusts have fixed magnitude and operates in an on-off mode. Numerical results are presented.

Key-Words: astrodynamics, artificial satellites, orbital dynamics, stochastic approach, transfer orbits.

1 Introduction

The main idea explored in the present paper is to study the orbital maneuvers required by a spacecraft that needs to perform a station keeping maneuver in order to remain with its orbital elements inside a region where it can achieve the goals of the mission. It is assumed that the orbit of the satellite is given, as well as a nominal orbit for this satellite that allows it to be useful for the planned activities. Then, it is necessary to maneuver this satellite from its current position to the nominal specified orbit.

The control available to perform this maneuver is the application of a low thrust to the satellite and the objective is to perform this maneuver with minimum fuel consumption. An optimal approach will be used, to allow the maximum possible savings. There is no time restriction involved here and the spacecraft can leave from any point in the initial orbit. In the present paper, the stochastic version of the projection of the gradient method is used. This version allows us to include the fact that the constraints do not need to be exactly satisfied (see reference [1] and [2]). This is done to realistically treat the numerical inaccuracies and/or flexibility in terms of tolerance in mission requirements, leading to situations where the final state is constrained to lie inside a given region, instead of having an exact value.

2 Review of Orbital Maneuvers

One of the most important works done in this fied is the one made by Hohmann [3]. He solved the problem of minimum ΔV transfers between two circular coplanar orbits. The Hohmann transfer would be generalized to the elliptic case (transfer between two coaxial elliptic orbits) by Marchal [4]. Smith [5] shows results for some other special cases, like coaxial and quasi-coaxial elliptic orbits, circular-elliptic orbits, two quasi-circular orbits. A numerical scheme to solve the transfer between two generic coplanar elliptic orbits is presented by Bender [6].

Another line of research studies the effects of the finite thrust, like the one used in the present paper, in the results obtained from the impulsive model. Zee [7] obtained analytical expressions for the extra fuel consumed to reach the same transfer and for the errors in the orbital elements and energy for a nominal maneuver (a real maneuver that uses the impulses calculated with the impulsive model).

Later, the literature studied the problem of a two-impulse transfer where the magnitude of the two impulses are fixed, like in Jin and Melton [8]; Jezewski and Mittleman [9].

The three-impulse concept was introduced in the literature by Hoelker and Silber [10] and Sthernfeld [11]. They showed that a bi-elliptical transfer between two circular orbits has a lower ΔV than the Hohmann transfer, for some combinations of initial and final orbits. After that, Ting [12] showed that the use of more than three impulses does not lower the ΔV, for impulsive maneuvers. Roth [13] obtained the minimum ΔV solution for a bi-elliptical transfer between two inclined orbits. Following the idea of more than two impulses, we have the work done by Prussing [14] that admits two or three

impulses; Prussing [15] that admits four impulses; Eckel [16] that admits N impulses.

Some other researchers worked on methods where the number of impulses is a free parameter, and not a value fixed in advance. It is the case of the papers made by Lion and Handelsman [17], Jezewski and Rosendaal [18], Gross and Prussing [19], Eckel [20] and Prussing and Chiu [21]. Most of the research done in this particular case is based on the "Primer-Vector" theory developed by Lawden [22], [23].

Another features that was introduced in the orbital maneuvers are the concepts of swing-by and gravitational capture. Those techniques use perturbations of a third body to increase or decrease the energy of the spacecraft. References [24] to [30] describe this problem in more details.

3 Definition of the Problem

The basic problem discussed in this paper is the problem of orbit transfer maneuvers. The objective of this problem is to modify the orbit of a given spacecraft by a small amount, that is compatible with the orbital corrections required by a spacecraft that needs to return to the nominal orbit that was affected by the perturbations [31, 32]. So, an initial and a final orbit around the Earth are completely specified. The problem is to find how to transfer the spacecraft between those two orbits in such way that the fuel consumed is minimum. There is no time restriction involved here and the spacecraft can leave and arrive at any point in the given initial and final orbits. The maneuver is performed with the use of an engine that is able to deliver a thrust with constant magnitude and variable direction. The mechanism, time and fuel consumption to change the direction of the thrust is not considered in this paper.

4 Model Used

The spacecraft is supposed to be in Keplerian motion controlled only by the thrusts, whenever they are active. This means that there are two types of motion:

i) A Keplerian orbit that is an orbit obtained by assuming that the Earth's gravity (assumed to be a point of mass) is the only force acting on the spacecraft. This motion occurs when the thrusts are not firing;

ii) The motion governed by two forces: the Earth's gravity field (also assumed to be a point of mass) and the force delivered by the thrusts. This motion occurs

during the time that the thrusts are firing.

The thrusts are assumed to have the following characteristics:

i) Fixed magnitude: The force generated by them is always of constant magnitude during the maneuver. The value of this constant is a free parameter (an input for the algorithm developed here) that can be high or low;

ii) Constant Ejection Velocity: Meaning that the velocity of the gases ejected from the thrusts is constant;

iii) Free angular motion: This means that the direction of the force given by the thrusts can be modified during the transfer. This direction can be specified by the angles u_1 and u_2, called pitch (the angle between the direction of the thrust and the perpendicular to the line Earth-spacecraft) and yaw (the angle with respect to the orbital plane);

iv) Operation in on-off mode: It means that intermediate states are not allowed. The thrusts are either at zero or maximum level all the time.

Several numbers of "thrusting arcs" (arcs with the thrusts active) are tested for each maneuver. Instead of time, the "range angle" (the angle between the radius vector of the spacecraft and an arbitrary reference line in the orbital plane) is used as the independent variable, as used by Biggs [33], [34].

5 Optimal Control Formulation

The minimum fuel spacecraft maneuver can be treated as a typical optimal control problem, formulated as follows:

Objective Function: Let M_f, the final mass of the vehicle, to be maximized with respect to the control $\underline{u}(.)$;

Subject to:

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, s); \qquad (1)$$
$$\underline{Ce}(\underline{x}, \underline{u}, s) = \underline{Ee}; \qquad (2)$$
$$\underline{Cd}(\underline{x}, \underline{u}, s) \leq \underline{Ed}; \qquad (3)$$
$$\underline{h}(\underline{x}(t_f), t_f) = \underline{Eh}, t_0 \text{ and } \underline{x}(t_0) \text{ given} \qquad (4)$$

where $\underline{x}$ is the state vector, $\underline{f}(.)$ is the right hand side of equations of motion, as in Biggs [34] and Prado and Rios-Neto [35]; s is the independent variable ($s_0 \leq s \leq s_f$), $\underline{Ce}(.)$ and $\underline{Cd}(.)$ are the algebraic dynamic constraints on state and control of dimensions m_e and m_d; $\underline{h}(.)$ are the boundary constraints of dimension m_h; and $\underline{Ee}, \underline{Ed}, \underline{Eh}$ error vectors satisfying:

$$|Ee_i| \le Ee_i^T, \quad i = 1, 2, 3, ..., m_e \quad (5)$$

$$|Ed_i| \le Ed_i^T, \quad i = 1, 2, 3, ..., m_d \quad (6)$$

$$|Eh_i| \le Eh_i^T, \quad i = 1, 2, 3, ..., m_h \quad (7)$$

where the fixed given values Ee_i^T, Ed_i^T, Eh_i^T, characterizes the region around zero within which errors are considered tolerable.

6   Mathematical Method

This approach is based on Optimal Control Theory. First order necessary conditions for a local minimum are used to obtain the adjoint equations and the Pontryagin's Maximum Principle to obtain the control angles at each range angle, leading to a "Two Point Boundary Value Problem" (TPBVP), where the difficulty is to find the initial values of the Lagrange multipliers. The treatment given here [34] is the hybrid approach of guessing a set of values, integrating numerically all the differential equations and then searching for a new set of values, based on a nonlinear programming algorithm. With this approach, the problem is again reduced to parametric optimization, as in the suboptimal method, with the difference that the angles' parameters are replaced by the initial values of the Lagrange multipliers, as variables to be optimized.

The method showed by Biggs [34] was used, where the "adjoint-control" transformation is performed and, instead of the initial values of Lagrange multipliers, one guesses control angles and their rates at the beginning of thrusting. With this, it is easier to find a good initial guess, and the convergence is faster. This hybrid approach has the advantage that, since the Lagrange multipliers remain constant during the "ballistic arcs", it is necessary to guess values of the control angles and its rates only for the first "burning arc". This transformation reduces very much the number of variables to be optimized and, as a consequence, the time of convergence.

7 Numerical Method

To solve the nonlinear programming problem, the stochastic version of the projection of the gradient method (Rios-Neto and Pinto [1]) was used.

Its general scheme is resumed in what follows:

Given a value $\overline{p}$ of the searched vector of parameters $\underline{p}$, from an initial guess or from an immediately previous iteration, a first order, direct search approach is adopted in a typical iteration to determine an approximate solution for the increment $\Delta\underline{p}$ in the problem:

Minimize: $J(\overline{p} + \Delta\underline{p})$ (10)

Subject to: $\underline{Ce}(\overline{p} + \Delta\underline{p}) = \alpha\underline{Ce}(\overline{p}) + \underline{Ee}$ (11)
$\underline{Cd}(\overline{p} + \Delta\underline{p}) = \beta\underline{Cd}(\overline{p}) + \underline{Ed}$ (12)

where $J(\underline{p})$ is the objective function; $\underline{Ce}(\underline{p})$ the equality constraints; $\underline{Cd}(\underline{p})$ the active inequality constraints at $\underline{p}$; and $0 \le \alpha < 1$, $0 \le \beta < 1$ are chosen close enough to one to lead to increments $\Delta\underline{p}$ of a first order of magnitude.

Linearized approximations are taken for the left hand sides of Equations (11) and (12) together with a stochastic interpretation for the errors $\underline{Ee}$ and $\underline{Ed}$, resulting in:

$$(\alpha - 1)\underline{Ce}(\overline{p}) = (d[\underline{Ce}(\overline{p})]/d\underline{p})\,\Delta\underline{p} + \underline{Ee} \quad (13)$$

$$(\beta - 1)\underline{Cd}(\overline{p}) = (d[\underline{Cd}(\overline{p})]/d\underline{p})\,\Delta\underline{p} + \underline{Ed} \quad (14)$$

where $\underline{Ed}$ and $\underline{Ee}$ are now assumed to be zero mean uniformly distributed errors, modeled as:

$$E[\underline{Ee}\,\underline{Ee}^T] = \text{diag}\,[\,e_i, i = 1,2,...,m_e]$$
$$E[\underline{Ed}\,\underline{Ed}^T] = \text{diag}\,[\,d_i, i = 1,2,...,m_d]$$

where $E[.]$ indicate the expected value of its argument.

The condition of Equation (10) is approximated by the following "a priori information":

$$-g.\nabla J^T(\overline{p}) = \Delta\underline{p} + \underline{n} \quad (15)$$

where $g \ge 0$ is to be adjusted to guarantee a first order of magnitude for the increment, that is, such that $\Delta\underline{p}$ is small enough to permit the use of a linearized representation of $J(\overline{p}+\Delta\underline{p})$; and $\underline{n}$ is taken as a zero mean uniformly distributed random vector, modelling the a priori searching error in the direction of the gradient $\nabla J(\overline{p})$, with:

$$E[\underline{n}\,\underline{n}^T] = \overline{P}$$

as its diagonal covariance matrix. The values of the variances in $\overline{P}$ are chosen such as to characterize an "adequate order of magnitude" for the dispersion of $\underline{n}$.

The diagonal form adopted is to model the assumption that it is not imposed any a priori correlation between the errors in the gradient components.

The simultaneous consideration of conditions of Equations (13), (14) and (15) characterize a problem of parameter estimation, which in a compact notation can be put as follows:

$$\overline{\overline{U}} = \underline{U} + \underline{n} \tag{16}$$

$$\overline{\underline{Y}} = \underline{MU} + \underline{V} \tag{17}$$

where $\overline{\overline{U}} \triangleq$ -g.$\nabla J^T(\overline{p})$ is the "a priori information";

$\underline{U} \triangleq \Delta \underline{p}$;

$\underline{Y} \triangleq [(\alpha - 1)\underline{Ce}^T(\overline{p}) : (\beta - 1)\underline{Cd}^T(\overline{p})]$ is the observation vector;

$\underline{M}^T \triangleq [(d(\underline{Ce}(\overline{p}))/d\underline{p})^T : (d(\underline{Cd}(\overline{p}))/d\underline{p})^T]$; $\underline{V}^T = [\underline{Ee}^T : \underline{Ed}^T]$.

Adopting a criterion of linear, minimum variance estimation, the optimal search increment can be determined using the classical Gauss-Markov estimator, which in Kalman form (e. g. Jazwinski [32]) gives:

$$\hat{\underline{U}} = \overline{\overline{U}} + \underline{K}(\underline{Y} - \underline{M}\overline{\overline{U}}) \tag{18}$$

$$\underline{P} = \overline{\underline{P}} - \underline{KM}\overline{\underline{P}} \tag{19}$$

$$\underline{K} = \overline{\underline{P}}\underline{M}^T(\underline{M}\overline{\underline{P}}\underline{M}^T + \underline{R})^{-1} \tag{20}$$

where $\overline{\underline{P}}$ is defined as before; $\underline{R} \triangleq E[\underline{VV}^T]$ = diag $[R_k, k = 1,2,...,m_e+m_d]$; and $\underline{P}$ has the meaning of being the covariance matrix of the errors in the components estimates of $\underline{U}$, i. e.:

$$\underline{P} = E[(\underline{U} - \hat{\underline{U}})(\underline{U} - \hat{\underline{U}})^T] \tag{21}$$

To build a numerical algorithm using the proposed procedure, the following types of iterations are considered:

(i) Initial phase of acquisition of constraints, when starting from a feasible point that satisfies the inequality constraints, the search is done to capture the equality constraints, including those inequality constraints that eventually became active along this phase;

(ii) Search of the minimum, when from a point that satisfies the constraints in the limits of the tolerable errors $\underline{V}$ in Equation (17), the search is done to take the objective function (Equation (10)) to get closer to the minimum; this search is conducted relaxing the order of magnitude of the error bounds around the constraints;

(iii) Restoration of the constraints, when from a point that resulted from a type (ii) iteration, the search is done to restore constraints satisfaction, within the limits imposed by the error $\underline{V}$ in Equation (17).

Rios-Neto and Pinto [1] suggest how to choose good values for the numerical parameters that must be different for each type of iteration.

8 - Simulations and Numerical Tests

To verify the algorithm proposed, two maneuvers of station keeping were simulated. These results were compared with the ones obtained by the deterministic version, without flexibility in constraint's satisfaction. Similar problems can be found in references [36], [37] and [38]. The first maneuver will occur with the data given in Table 1. The thrust level is 2.0 N. Table 2 shows the errors allowed in the final Keplerian elements of the orbit.

Table 1 - Data for Maneuver 1

Orbits	Initial	Final
Semi-major axis	7060.00	7090.00
Eccentricity	0.03	0.01
Inclination (degrees)	3.00	3.00
Ascending Node (degrees)	0.00	Free
Argument of perigee (degrees)	0.00	Free
Mean Anomaly (degrees)	0.00	Free

Table 2 - Errors Allowed for Final Keplerian Elements

Semi-major axis	1.0 Km
Eccentricity	0.005
Inclination	0.001 deg

The choice of the number of "burning arcs" was done for several different values for the same maneuver, in order to obtain some information about the importance of this parameter.

The consumptions found are showed in Table 3, as well as comparisons with deterministic methods, that we also simulated for comparison..

Table 3 - Fuel Expenditure Comparisons (kg)

Approach	Stochastic	Deterministic
2 Arcs	0.201	0.203
3 Arcs	0.200	0.201
4 Arcs	0.198	0.199
5 Arcs	0.197	0.198
6 Arcs	0.196	0.197
7 Arcs	0.195	0.196
8 Arcs	0.194	0.196

The second maneuver will occur with the data given in Table 4. Now, only the inclination is changed. The thrust level is again 2.0 N. Table 5 shows the errors allowed in the final Keplerian elements of the orbit.

Table 4 - Data for Transfer Maneuver 2

Orbits	Initial	Final
Semi-major axis	8000.00	8000.00
Eccentricity	0.02	0.02
Inclination (degrees)	3.00	2.50
Ascending Node (degrees)	0.00	Free
Argument of perigee (degrees)	0.00	Free
Mean Anomaly (degrees)	0.00	Free

Table 5 - Errors Allowed for Final Keplerian Elements

Semi-major axis	0.50 Km
Eccentricity	0.005
Inclination	0.001 deg

The choice of the number of "burning arcs" was done for several different values, in the same way that was made in the first maneuver.

The consumptions found are showed in Table 6, as well as comparisons with deterministic methods.

Table 6 - Fuel Expenditure Comparisons (kg)

Approach	Stochastic	Deterministic
2 Arcs	0.130	0.132
3 Arcs	0.128	0.130
4 Arcs	0.127	0.129
5 Arcs	0.127	0.128
6 Arcs	0.127	0.128
7 Arcs	0.127	0.128
8 Arcs	0.127	0.128

9 - Conclusions

Optimal control was explored to generate an algorithm to obtain solutions for the problem of minimum fuel consumption to make orbital maneuvers of a satellite that needs to return to its nominal orbit after deviations caused by perturbations forces.

This problem was considered taking into account the accuracy tolerance in the constraint's satisfaction using the new nonlinear programming algorithm proposed by Rios-Neto and Pinto [1].

The results showed that some fuel can be saved by exploring tolerable errors allowed for constraint's satisfaction. The amount saved is not negligible.

It is also possible to see that the increase of the number of burning arcs can decrease the total fuel expenditure. This happens because this increase is accompanied by an increase of the number of variables available for the optimization technique. Since the maneuvers have small amplitudes, the increasing of the number of arcs has a limit in the possible savings and the fuel consumption goes to a constant value after the number of arcs reach a certain value.

Acknowledgments

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