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On the formulation and solution of the isochronal synchronization stability problem in delay-coupled complex networks

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We present a new framework to the formulation of the problem of isochronal synchronization for networks of delay-coupled oscillators. Using a linear transformation to change coordinates of the network state vector, this method allows straightforward definition of the error system, which is a critical step in the formulation of the synchronization problem. The synchronization problem is then solved on the basis of Lyapunov-Krasovskii theorem. Following this approach, we show how the error system can be defined such that its dimension can be the same as (or smaller than) that of the network state vector. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4753921]

Isochronal synchronization is amongst the most intriguing collective behaviors observed in coupled chaotic oscillators and networks. The oscillators' dynamics behave identically in time, despite of time-delays in the coupling signals. Although reported in numerical simulations, experimental setups, and analytical studies, there are several open problems within the topic. In particular, there have been several attemps to reduce the restrictiveness of problem formulations and their respective solutions, especially those in which feedback controllers are used. Formulations commonly consider that the network nodes achieve synchronization by following a target reference signal, which ultimately jeopardizes the applicability of resulting stability criteria to real-world network problems. Another actual difficulty is the high-dimensionality of the resulting stability criteria, which makes stability evaluation costly. Towards the improvement of existing frameworks and the extension of their practical scope, a new framework is proposed, which allows simple problem formulation and privileges the study of network synchronization problems in the case when synchronicity emerges solely as a result of the interplay among the nodes' dynamics. As a complement, a general stability criterion is derived for isochronal synchronization, based on the Lyapunov-Krasovskii theorem. Given a network of chaotic oscillators, it is shown how to check for stability of isochronal synchronization by simply feeding a matrix inequality with some accessible parameters of the network. Examples of application of the criteria, in the form of stability functions over the network parameter space are presented for *k*-cycle networks to illustrate the effectiveness and feasibility of the analytical results.

I. INTRODUCTION

Isochronal synchronization is a physical phenomenon in which oscillators coupled with time-delay achieve zero-lag identical behavior in time, and it has attracted attention in the last few years.[1–12] For example, picture the ensembles of neurons located in different regions of the brain, which fire together in the performance of cognitive acts and constitute an actual topic of research of neurology, among others.[13–20] As remotely located, it is natural to imagine that the synchronization among such ensembles might be influenced by the coupling delay, i.e., the time of propagation of messages transiting from one region of the brain to the other. Surprisingly under some circumstances, the oscillators' synchronicity establishes with zero lag even in the presence of coupling delays.[7]

In general, sources of coupling delay are well-known in natural and engineering network systems. More importantly, the distance among the nodes and the speed of propagation of coupling signals through physical media can be mentioned. However, the reasons why isochronal (zero-lag) synchronization establishes in the presence of coupling delays are still intriguing and subject to controversial debate.[21] Recent studies have unveiled some of its subjacent mechanisms, both within the context of pairs of oscillators[1–8] and networks.[9–12] In this context, the relation between feedback and coupling times was shown to have particular influence upon its emergence and maintenance.[6] Meanwhile, other features of the coupling setup, such as network topology and coupling strength were shown to be decisive, as observed previously for networks with non-delayed couplings.[22]

Towards the solution of the network synchronization problem, control techniques are generally used to drive the oscillators, clusters or networks to different forms of synchrony, including isochronal.[7,8,10–12,23–28] In some cases, this is done by assuming an external input reference signal in the control loop, such that zero-lag synchronization is achieved on the basis of a common target trajectory.[23–27] Within such framework, analytic criteria were derived so that parameters of the network are adjusted (e.g., feedback matrices), the

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network is rendered synchronous and synchronization is rendered stable.

Although very interesting from the theoretical viewpoint, such frameworks might hold too restrictive requirements for practical applications. The main reason is that availability of a common target signal to simultaneously drive all of the network nodes is a stringent requirement which is recognized to be not achievable in most network configurations. Note that the physical separation, which is the main cause of delays, is a natural result of the distributed nature of network systems and the distance among its nodes, such that transmission delay is almost surely present in practical applications. Thus, any target reference signal would be subject to delay as well, such that simultaneity in the signal reception is lost.

In this context, a more realistic approach would consider a network setup in which isochronal synchronization is induced solely by the interplay among the nodes in the feedback control loop, as in Refs. 10 and 11. However, there are difficulties associated with such approach, since the resulting stability criteria may result in quite high-dimensional matrix inequalities, which tend to make the stability investigation harder and its results more conservative. To overcome these problems, a new framework is proposed to the formulation of the network synchronization problem. It is aimed at generating the synchronization error system, whose stability means that of synchronization of the original network. Following this approach, the process of definition of the error equations is revealed straightforward, leading to simpler problem formulation and to more general results. To illustrate the framework, a stability criterion for isochronal network synchronization is developed on its basis, considering the Lyapunov-Krasovskii theorem.[29]

As a consequence, analytic stability criteria are derived, which require a few accessible parameters of the network setup to be checked and allow the design and test of coupling configurations that render the network synchronized. The results are aimed at providing a general framework to the evaluation of network synchronization stability and serving as backbone for upcoming technological applications of isochronal synchronization, such that ones in communication.[1,2] The proposed problem formulation is argued to open new possibilities for the derivation of more general and less restrictive stability criteria in the context of isochronal synchronization of networks of delay-coupled chaotic oscillators. Numerical results illustrate the effectiveness of the analytical developments, as networks with up to a hundred oscillators are systematically checked for stability of isochronal synchronization.

The paper is organized as follows: Sec. II reviews the usual problem formulation in the literature and introduce a new framework; Sec. III uses the new formulation to derive an isochronal synchronization stability criterion on the basis of the Lyapunov-Krasovskii theorem; Sec. IV presents examples of application of the criterion for the evaluation of stability functions in the parameter spaces of k-cycle networks of Lorenz[30] and Rössler[31] oscillators; Sec. V discusses the main results and, finally, Sec. VI brings final remarks. The assumptions considered in the problem formulation are that (i) the nodes are all identical and chaotic, such that the trajectories are trapped into a set Ω that contains the chaotic attractor and (ii) the coupling delay is constant and equal for every link in the network.

II. FORMULATION OF THE NETWORK SYNCHRONIZATION PROBLEM

As pointed in Refs. 10 and 11, synchronization of delay-coupled oscillators is more widely studied for the case when the coupling term appearing between nodes i and j is given in terms of $x_i(t-\tau) - x_j(t-\tau)$, in which both states feature time-delay. However, such abstraction does not hold in most practical situations, where self-feedback is not present. Thus, a more realistic approach would consider the coupling term of the node i as given by

$$x_i(t) - x_j(t-\tau), \tag{1}$$

in which only the state variable that traveled from node j to node i is affected by transmission delay.[10]

The formulation of the synchronization problem under this latter form of coupling involves a remarkable difference: the coupling term does not always vanish in the synchronization manifold as it is does in the former case. This fact is explored in Ref. 10, where it is concluded that the existence of the synchronization manifold in this case depends on symmetry conditions of the coupling delays and coupling matrices. Once satisfied, such conditions would guarantee that the synchronization manifold exists.

Taking into account such and towards the formulation for the problem of isochronal synchronization, consider the equations of the *ith* oscillator as

$$\dot{x}_i(t) = Ax_i(t) + g(x_i(t)) - \frac{c}{G_{ii}} \sum_{j=1}^{N} G_{ij}\Gamma x_j(t-\tau_{ij}), \tag{2}$$

where $A \in \mathbb{R}^n \times \mathbb{R}^n$ is a constant matrix, $g: \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz continuous vector function, $c \in \mathbb{R}$ is an scalar coupling term, $G \in \mathbb{R}^N \times \mathbb{R}^N$ symmetric and with zero row sum is the Laplacian matrix of the network, which is assumed connected, and $\Gamma \in \mathbb{R}^n \times \mathbb{R}^n$ is the node inner coupling matrix. For the existence of the synchronization manifold, it is assumed that $\tau_{ij} = \tau$ for $i \neq j$ and, $\tau_{ij} = 0$ for $i = j$, for direct self-coupling. As a result, Eq. (2) can be rewritten as

$$\dot{x}_i(t) = (A - c\Gamma)x_i(t) + g(x_i(t)) - \frac{c}{G_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{N} G_{ij}\Gamma x_j(t-\tau), \tag{3}$$

and G_{ij} are entries of the Laplacian matrix G and $\frac{c}{G_{ii}}$ guarantees the node balance condition, which is also necessary for the existence of the synchronization manifold. As the state vectors $x_1(t), x_2(t), \ldots, x_N(t)$ are collected into the network state vector $x(t) \in \mathbb{R}^{Nn}$, given by

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix}, \tag{4}$$

the equations of the delay-coupled network can be written in the compact form

$$\dot{x}(t) = I_N \otimes (A - c\Gamma)x(t) + \Phi(x(t)) - c\left(D^{-1}A_d \otimes \Gamma\right)x(t-\tau), \quad (5)$$

where I_N is an N-dimensional identity matrix, $\otimes$ is the Kronecker product,

$$\Phi(x(t)) = \begin{pmatrix} g(x_1(t)) \\ \vdots \\ g(x_N(t)) \end{pmatrix}. \quad (6)$$

Additionally, A_d is the network adjacency matrix that assigns $A_{d_{ij}} = A_{d_{ji}} = -1$ if nodes i and j are connected, $A_{d_{ij}} = A_{d_{ji}} = 0$ otherwise and, $A_{d_{ii}} = 0$, and $G = D - A_d$. Note that, for a connected network, D is nonsingular and Eq. (5) holds.

At this point, the dynamical equations of the network are available and the formulation of the synchronization problem requires the definition of an error vector function $e(t)$, such that $\|e(t)\| \to 0$ as $t \to \infty$ implies $x_i(t) = x_j(t)$ for every $i, j = 1, ..., N$. Towards that end, define

$$e(t) = x_A(t) - x_B(t), \quad (7)$$

where $x_A(t) = T_A x(t)$, $x_B(t) = T_B x(t)$ and $T_A, T_B \in \mathbb{R}^{N-1} \times \mathbb{R}^N$ are chosen as

$$T_A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \otimes I_n; \quad T_B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \otimes I_n, \quad (8)$$

where I_n stands for the n-dimensional identity matrix,

$$T_A\dot{x}(t) = T_A I_N \otimes (A - c\Gamma)x(t) + T_A\Phi(x(t)) - cT_A\left(D^{-1}A_d \otimes \Gamma\right)x(t-\tau), \quad (9)$$

$$T_B\dot{x}(t) = T_B I_N \otimes (A - c\Gamma)x(t) + T_B\Phi(x(t)) - cT_B\left(D^{-1}A_d \otimes \Gamma\right)x(t-\tau), \quad (10)$$

and $\dot{e}(t)$ can be written as

$$\begin{aligned} \dot{e}(t) &= T_A\dot{x}(t) - T_B\dot{x}(t) \\ &= (T_A - T_B)\dot{x}(t) \\ &= (T_A - T_B)I_N \otimes (A - c\Gamma)x(t) + \Phi(x_A(t)) \\ &\quad - \Phi(x_B(t)) - c(T_A - T_B)\left(D^{-1}A_d \otimes \Gamma\right)x(t-\tau). \end{aligned} \quad (11)$$

Introducing the state-dependent matrix $M(x(t))e(t) = \Phi(x_A(t)) - \Phi(x_B(t))$, for which it is noticeable that $T_A\Phi(x(t)) = \Phi(T_A x(t)) = \Phi(x_A(t))$, $T_B\Phi(x(t)) = \Phi(T_B x(t)) = \Phi(x_B(t))$, the nonlinear part of the individual systems' dynamics is written in terms of the error variables $e(t)$. At this point, note that system (11) depends both on the error states $e(t)$ and the network states $x(t)$, which is undesirable since the synchronization stability evaluation must be performed in the error variables alone. Thus, considering Eqs. (7) and (8), one can rewrite systems (11) in terms of the error variables $e(t)$ by designing an adequate matrix of coefficients E such that

$$-c(T_A - T_B)\left(D^{-1}A_d \otimes \Gamma\right)x(t) = E(T_A - T_B)x(t), \quad (12)$$

and it follows that $E(T_A - T_B) = -c(T_A - T_B)(D^{-1}A_d \otimes \Gamma)$, as $(T_A - T_B)x(t) = e(t)$. As a result, the error system can be rewritten as

$$\dot{e}(t) = I_{N-1} \otimes (A - c\Gamma)e(t) + M(x(t))e(t) + Ee(t-\tau), \quad (13)$$

where all state variables are given in terms of $e(t)$. Note that $\|e(t)\| = 0$ implies and is implied by $x_1(t) = x_2(t) = ... = x_N(t)$, i.e., the stability of network synchronization requires that the error system asymptotically establishes at the trivial fixed point of system (13), and vice-versa.

Towards the determination of conditions for synchronization stability, consider the identity $e(t-\tau) = e(t) - \int_{t-\tau}^{t} \dot{e}(\theta)d\theta$,[32] such that the error equations can be rewritten as

$$\dot{e}(t) = I_{N-1} \otimes (A - c\Gamma)e(t) + M(x(t))e(t) + E\left(e(t) - \int_{t-\tau}^{t} \dot{e}(\theta)d\theta\right), \quad (14)$$

and, one step ahead,

$$\begin{aligned} \dot{e}(t) &= [I_{N-1} \otimes (A - c\Gamma) + M(x(t)) + E]e(t) \\ &\quad - E\int_{t-\tau}^{t} [I_{N-1} \otimes (A - c\Gamma)e(\theta) + M(x(t))e(\theta) \\ &\quad + Ee(\theta - \tau)]d\theta. \end{aligned} \quad (15)$$

Notice that, according to this formulation, the definition of the error system dismisses the use of reference signals, and synchronization, if it establishes, is a result of the interaction among the dynamics of the network nodes.

Considering these developments, Sec. III presents a solution for the synchronization problem under this formulation. A stability criterion for the trivial fixed point of the error system (15) is derived, by means of the Lyapunov-Krasovskii stability theorem.[29]

III. STABILITY CRITERION FOR ISOCHRONAL SYNCHRONIZATION IN COMPLEX NETWORKS

In the context of delay-coupled complex networks, isochronal synchronization can be defined as zero-lag synchronization among all the network nodes. In other words, consider N oscillators, whose states are given by $x_i(t) \in \mathbb{R}^n$, $i = 1, 2, ..., N$, coupled with time-delay τ, in a network configuration. Recall that isochronal synchronization means the identity $x_1(t) = x_2(t) = \ldots = x_N(t)$ despite time-delay is present in the coupling, such that the oscillators' dynamics achieve zero-lag synchronicity.

At this point, the establishment of conditions for stability of system (15) about the origin may be investigated using the Lyapunov-Krasovskii theorem.[29] Towards this end, consider a Lyapunov-Krasovskii functional in the form

$$V(e(t)) = V_1(e(t)) + V_2(e(t)), \tag{16}$$

where

$$V_1(e(t)) = e^T(t)Pe(t), \tag{17}$$

$$V_2(e(t)) = \frac{2\varepsilon}{\tau}\int_{-\tau}^{0}\left[\int_{t+\theta}^{t} e^T(s)\Upsilon^T\Upsilon e(s)ds + \int_{t+\theta-\tau}^{t} e^T(s)E^TEe(s)ds\right]d\theta. \tag{18}$$

and

$$\Upsilon = I_{N-1}\otimes(A - c\Gamma + L) \tag{19}$$

for some matrix $P \in \mathbb{R}^{(N-1)n}\times\mathbb{R}^{(N-1)n}$, $P = P^T > 0$ and some constant matrix $L \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying $L - M(x(t)) > 0$ for all $x(t) \in \Omega$. Alternatively, for local stability analysis, one can choose L as an n-dimensional null matrix.

To carry the stability analysis, the time-derivative of the function (16) has to be evaluated in the trajectories of the error system (15). As the time-derivative of the function $V(e(t))$ is given by

$$\dot{V}(e(t)) = \dot{V}_1(e(t)) + \dot{V}_2(e(t)), \tag{20}$$

where

$$\begin{aligned}\dot{V}_1(e(t)) &= \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t)\\ &= e^T(t)[I_{N-1}\otimes(A - c\Gamma) + M(x(t)) + E]^TPe(t)\\ &\quad + e^T(t)P[I_{N-1}\otimes(A - c\Gamma) + M(x(t)) + E]e(t)\\ &\quad - 2e^T(t)PE\int_{t-\tau}^{t}[I_{N-1}\otimes(A - c\Gamma)e(\theta)\\ &\quad + M(x(t))e(\theta) + Ee(\theta-\tau)]d\theta,\end{aligned} \tag{21}$$

$$\begin{aligned}\dot{V}_2(e(t)) &= \frac{2\varepsilon}{\tau}\int_{-\tau}^{0}[e^T(t)\Upsilon^T\Upsilon e(t) + e^T(t)E^TEe(t)]d\theta\\ &\quad - \frac{2\varepsilon}{\tau}\int_{-\tau}^{0}[e^T(t+\theta)\Upsilon^T\Upsilon e(t+\theta)\\ &\quad + e^T(t+\theta-\tau)E^TEe(t+\theta-\tau)]d\theta,\end{aligned} \tag{22}$$

then, by evaluating the first integral in Eq. (22), the equation can be rewritten as

$$\begin{aligned}\dot{V}_2(e(t)) &= 2\varepsilon[e^T(t)\Upsilon^T\Upsilon e(t) + e^T(t)E^TEe(t)]\\ &\quad - \frac{2\varepsilon}{\tau}\int_{-\tau}^{0}[e^T(t+\theta)\Upsilon^T\Upsilon e(t+\theta)\\ &\quad + e^T(t+\theta-\tau)E^TEe(t+\theta-\tau)]d\theta.\end{aligned} \tag{23}$$

Concerning Eq. (21), as one considers $A^TB + B^TA \le \varepsilon A^TA + \varepsilon^{-1}B^TB$, it can be rewritten as

$$\begin{aligned}\dot{V}_1(e(t)) &\le -e^T(t)Qe(t) + e^T(t)E^TP^TPEe(t) + \Delta^T\Delta \le \ldots\\ &\le -e^T(t)Qe(t) + \frac{\tau}{\varepsilon}e^T(t)E^TP^TPEe(t) + \frac{\varepsilon}{\tau}\Delta^T\Delta,\end{aligned} \tag{24}$$

where

$$\begin{aligned}-Q &= [I_{N-1}\otimes(A - c\Gamma + L) + E]^TP\\ &\quad + P[I_{N-1}\otimes(A - c\Gamma + L) + E],\\ &= [\Upsilon + E]^TP + P[\Upsilon + E],\end{aligned} \tag{25}$$

and, further,

$$\Delta = \int_{t-\tau}^{t}[I_{N-1}\otimes(A - c\Gamma)e(\theta) + M(x(t)) + Ee(\theta-\tau)]d\theta. \tag{26}$$

It follows that

$$\begin{aligned}\Delta^T\Delta &\le \int_{-\tau}^{0}[\Upsilon e(t+\theta) + Ee(t+\theta-\tau)]^T\\ &\quad \cdot[\Upsilon e(t+\theta) + Ee(t+\theta-\tau)]d\theta.\\ &\le 2\int_{-\tau}^{0}[e^T(t+\theta)\Upsilon^T\Upsilon e(t+\theta)\\ &\quad + e^T(t+\theta-\tau)E^TEe(t+\theta-\tau)]d\theta.\end{aligned} \tag{27}$$

Finally, taking Eqs. (23)–(27) into Eq. (20) yields

$$\begin{aligned}\dot{V}(e(t)) &\le -e^T(t)Qe(t) + \frac{\tau}{\varepsilon}e^T(t)E^TP^2Ee(t)\\ &\quad + \frac{2\varepsilon}{\tau}\int_{-\tau}^{0}[e^T(t+\theta)\Upsilon^T\Upsilon e(t+\theta)\\ &\quad + e^T(t+\theta-\tau)E^TEe(t+\theta-\tau)]d\theta\\ &\quad + 2\varepsilon[e^T(t)\Upsilon^T\Upsilon e(t) + e^T(t)E^TEe(t)]\\ &\quad - \frac{2\varepsilon}{\tau}\int_{-\tau}^{0}[e^T(t+\theta)\Upsilon^T\Upsilon e(t+\theta)\\ &\quad + e^T(t+\theta-\tau)E^TEe(t+\theta-\tau)]d\theta.\end{aligned} \tag{28}$$

Simplifying the latter expression, one obtains

$$V(e(t)) \le -e^T(t)\left[Q - 2\varepsilon\Upsilon^T\Upsilon - 2\varepsilon E^TE - \frac{\tau}{\varepsilon}E^TP^2E\right]e(t). \tag{29}$$

Considering the inequality (29), the following result can be established:

Theorem (isochronal synchronization of delay-coupled oscillators in complex networks): *If there exists a constant matrix* $P = P^T > 0$ *and a positive constant* $\varepsilon > 0$ *such that*

$$W = Q - 2\varepsilon\Upsilon^T\Upsilon - 2\varepsilon E^T E - \frac{\tau}{\varepsilon}E^T P^2 E > 0 \qquad (30)$$

holds for a matrix $Q = Q^T > 0$*, where Q is given by Eq.* (25) *and* Υ *is given by Eq.* (19)*, then the delay-coupled network whose error system is given by Eq.* (13) *achieves isochronal synchronization for coupling delay* τ.

To illustrate the effectiveness of the analytical results, Sec. IV presents the results of computational evaluation of the synchronization stability criterion for *k*-cycle networks of chaotic oscillators. Further, the results show how synchronization stability can be systematically evaluated to a whole class of networks by using the inequality (30) to define a stability function over the network parameter space.

IV. EXAMPLES OF APPLICATION

Figures 1–3 present local stability maps for *k*-cycle networks, considering the Lorenz equations (Figure 1) and the Rössler equations (Figures 2 and 3). Such maps show the value of the minimum eingenvalue of the matrix $-W$ from inequality (30). Note that positive values of $\lambda_{\min}(-W)$ imply synchronization stability, while nothing concerning stability can be inferred from null and negative values, according to the theory of stability of motion by Lyapunov. It is interesting to note that the stability maps allow the visualization of the direction of loss of stability within the network parameter space. In the cases plotted below, loss of stability is induced as (i) number of oscillators increase, (ii) the number of links decrease, and (iii) as coupling delay increases. Considering the stability maps, a multidimensional evaluation of stability within the network parameter space becomes possible by means of evaluation of the matrix inequality (30).

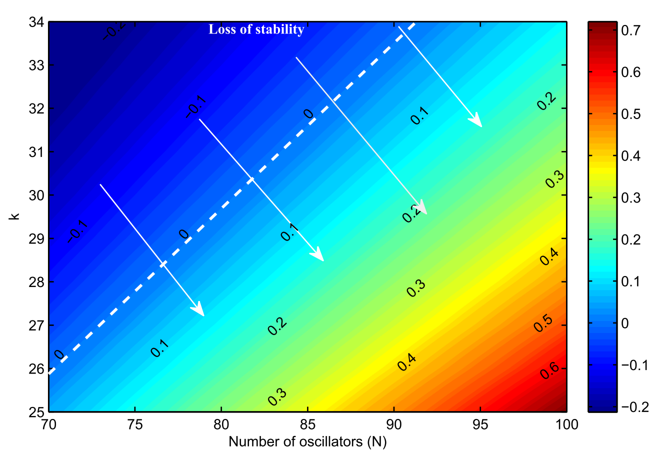

FIG. 1. Evaluation of synchronization stability of *k*-cycle networks of delay-coupled Lorenz oscillators,[30] over the set $N \times k$ of the network parameter space. In the stable region, $\lambda_{\max}(-W) < 0$; note that sync stability is lost as the number of oscillators increases and/or the number of links decreases, as indicated by arrows.

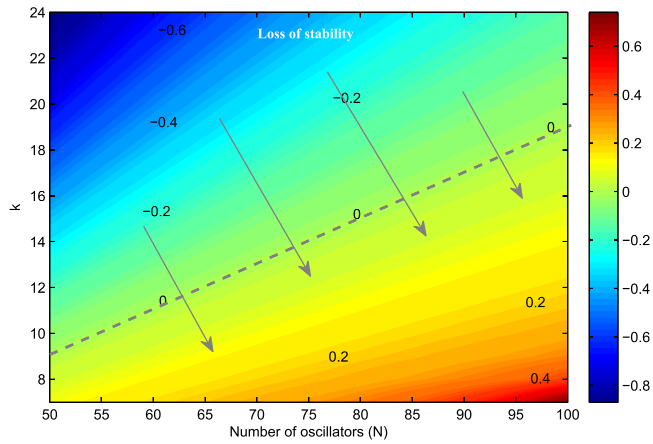

FIG. 2. Evaluation of synchronization stability of *k*-cycle networks of delay-coupled Rössler oscillators,[31] over the set $N \times k$ of the network parameter space. In the stable region, $\lambda_{\max}(-W) < 0$; note that sync stability is lost as the number of oscillators increases and/or the number of links decreases, as indicated by arrows.

A similar systematic approach can be used to trace such map for other kinds of network, as other parameters are chosen. To make the visual analysis simple, it is interesting note that a 2− *dimensional* parameter meshgrid is generated such that the influence of such parameters in the network stability can be properly evaluated in the resulting graphics. Further analysis and insights into the nature of network stability are possible as other parameters are considered in the generation of the domain for evaluation of other similar stability functions resulting from the analytical stability results presented in this paper.

V. DISCUSSION

The analytical results allow the evaluation of the stability condition for isochronal synchronization in undirected networks of delay-coupled oscillators with arbitrary topology by collecting accessible parameters of the network setup. It is assumed that nodes are all identical and that the coupling

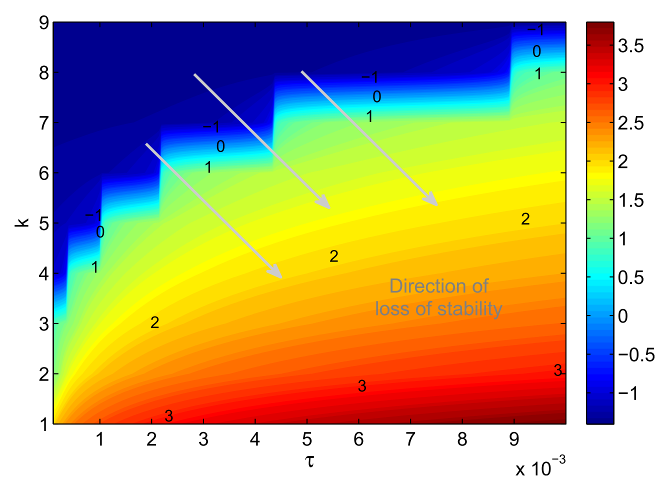

FIG. 3. Stability function over the set $\tau \times k$ of the parameter space of a *k*-cycle network of 20 Rössler oscillators.[31] Note that sync stability is lost as the number of links decreases and the time-delay increases.

delay is constant and equal for every link in the network. In practical terms, while the assumption on identical nodes can be relaxed without major trouble, due to inherent robustness characteristics of isochronal synchronization observed in numerical and experimental studies,[1,6] the assumption that coupling delays between any two connected nodes are equal is recognized to be limiting to some extent. Nevertheless, in some occasions, electronic compensation of time-delays may be used in practice to artificially equate differences in delay magnitude arising in practice.

In a broader context, the analytical results are consistent with known aspects of the physical phenomenon of isochronal synchronization, as described in the literature.[1–12] At first, the form that the delay term appears in the stability criterion suggests that stability is degraded for larger values of delay. Although this does not always hold, it is recognized that the qualitative behavior of Delay Differential Equations (DDEs) does change as delays increase, generally towards the loss of stability, due to the occurrence of Hopf bifurcations which induces oscillatory and unstable behavior.[33]

Besides, as shown in Figures 1 and 2, network are less likely to achieve isochronal synchronization as the number of oscillators is increased and the number of links among oscillators is decreased. This can be inferred from the direction of the loss of stability within the set $N \times k$. Stability is weakened in the direction perpendicular to that of the white dashed line, corresponding to an increasing number of nodes and a decreasing number of links. This observation agrees with those concluded for networks without delay which consider the eingenvalue gap of the network Laplacian matrix.[22]

Further and more general conclusions can be obtained about more general networks, such as small-world or scale-free networks using this methodology. Besides, considering the requirements for its applications, the presented approach can be used to evaluate the stability of synchronization in networks of limit-cycle oscillators, including general studies based on the network parameter space, as the ones presented in the previous section.

VI. FINAL REMARKS

The major finding of this study is a new framework under which the problem of isochronal network synchronization can be formulated in a simple and physically meaningful view. In addition, a sufficient stability condition for isochronal synchronization in delay-coupled networks of chaotic oscillators was derived and presented. From a simple problem formulation, the mathematical equations of the synchronization error system are obtained; the stability of the trivial fixed point of the error system is then studied on the basis of Lyapunov-Krasovskii theorem. Tests considering complex networks of delay-coupled Lorenz or Rössler oscillators illustrate the potential of the analytic results. The stability problem of isochronal synchronization in delay-coupled networks was formulated and solved in the form of a straightforward stability criterion that requires at most some computation to be checked and allows network isochronal synchronization to be evaluated systematically over the network parameter space.

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APPENDIX: PICKING UP THE PARAMETERS FOR THE APPLICATION OF THE STABILITY CRITERION

Consider a 2-cycle network of $N = 5$ oscillators, such that

$$G = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

and with coupling constant c, inner coupling matrix Γ and coupling delay τ, such that the network equations for the ith node are given by

$$\dot{x}_i(t) = Ax_i(t) + g(x_i(t)) - \frac{c}{G_{ii}} \sum_{j=1}^{N} G_{ij} \Gamma x_j(t - \tau_{ij}),$$

$\tau_{ij} = \tau$ for $i \neq j$ and $\tau_{ij} = 0$ otherwise. Suppose that the individual dynamics of the network nodes are given by the Rössler equations[31]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b + x_1 x_3 \end{bmatrix},$$

such that the matrix of linear terms A and the nonlinear function $g(.)$ appearing in the equation of the ith node are given by

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}; \quad g(x(t)) = \begin{bmatrix} 0 \\ 0 \\ b + x_1 x_3 \end{bmatrix}.$$

Further, choose a matrix L such that $L - M(x(t)) > 0$ for $x(t) \in \Omega$ and choose $T_A = I_N \otimes I_n$ and

$$T_B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \otimes I_n,$$

such that $e_1 = x_1 - x_2$, $e_2 = x_2 - x_3$, $e_3 = x_3 - x_4$, $e_4 = x_4 - x_5$ and $e_5 = x_5 - x_1$. As a result, a matrix E can be obtained such that

$$-c(T_A - T_B)\left(D^{-1}A_d \otimes \Gamma\right) = E(T_A - T_B)$$

is satisfied. Considering the error variables, E can be chosen as

$$E = c\begin{bmatrix} -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} \end{bmatrix} \otimes \Gamma.$$

Finally, choosing some $P > 0$ and $\varepsilon > 0$ with appropriate dimensions, the elements for the evaluation of stability are available. Further, as one defines a stability function using the inequality (30), the stability of isochronal synchronization over a chosen subset of the network parameter space can be evaluated.

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