HIEMENZ FLOW IN POROUS MEDIUM WITH HEAT EXCHANGE AND NEGLIGIBLE THERMAL EXPANSION

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Abstract. *The viscous fluid motion generated by a two-dimensional impinging flow on a flat plane is called a Hiemenz flow. By using variables and coordinate transformations and the boundary-layer approximation one can reduce the Navier-Stokes equations into a third-order ordinary differential equation that can be solved using numerical methods. In the present work, we extend Hiemenz problem into a flow inside a low porosity porous medium with the existence of a intense interphase heat exchange. The same classical transformations are applied to obtain the governing equations for mass and energy transport inside the porous matrix, for both gas phase and solid phase. Using the asymptotic expansion method, we are able to obtain quasi-analytical profiles for temperature and velocity for the flowing gas, and the temperature profile for the porous matrix. Two different regions are observed for the problem: one corresponding to the Darcy flow and that scales spatially with the solid thermal conductivity, and an inner region close to the wall, that take into account viscous effects, and scales spatially with the gas thermal conductivity. The variable change is chosen in such a way that for very large permeabilities, the model is able to recuperate the equations of the classical Hiemenz flow and for very small permeabilities, it describes the Darcy flow.*

Keywords: *hiemenz flow, porous medium, stagnation-point flow*

1. INTRODUCTION

The two-dimensional impinging flow over a flat surface was examined in the beginning of the last century (Hiemenz, 1911), in a work that demonstrated that the Navier-Stokes equations could be reduced into a third-order ordinary differential equation. However, since this equation present non-linear terms, it could only be solved by numerical methods.

Due to the possibility of practical applications, the interest in the geometry studied by Hiemenz have grown immensely in the last decades. For example, the Hiemenz flow is important in the study of a flow established close to an aircraft nose, when the analysis of the boundary layer effects become relevant, such as heat transfer and viscous attachment.

The interest in the study of flows established inside porous media have also grown considerably, since this geometry arises in many different systems, ranging from natural to technological, man-manufactured, ones. For example, cooling devices for electronic components were developed using porous materials (Cao and Ponnappan, 2008). Petroleum wells and underground aquifers are other examples of porous media. The study of fluid flow in those systems is of a major importance to the world economy.

In the present work, we study the geometry established from a stagnation-point flow inside a porous medium with a constant wall temperature. Such system may be though as a cooling device for electronic components, because of the efficient heat distribution of the porous matrix. An impinging flow configuration was analyzed analytically before (Wu et al., 2005), but the authors did not considered heat exchange. Their motivation lied on the study of the air flow beneath a sky or a snowboard, in which the snow is a porous layer.

The influence of the porosity, when one considers the existence of heat exchange, was analyzed numerically before (Attia, 2007). However, the numerical approach presents a difficulty, due to the fact that when one consider the existence of the heat exchange, different characteristic scales arises, and the numerical code may be non-efficient, losing information in the small scales, or even not being able to converge, except when one refines the grid in the region close to the wall.

An analytical procedure using the asymptotic theory is proposed in the present work. The characteristic length scales are identified and the results are obtained for each case. A high value for the interphase heat exchange is considered.

The results obtained analytically are consistent with numerical analysis existent in literature, at proper limits (Attia, 2007), showing that the methodology is valid.

2. MATHEMATICAL FORMULATION

In the proposed model we use the conservation equations under a modified variable transformation, proper to the stagnation-point flow (Schlichting, 1968).

The non-dimensional variables are given by $u \equiv \bar{u}/\bar{v}_\infty$, $\varrho \equiv \rho/\rho_\infty = 1$, $v \equiv \bar{v}/\bar{v}_\infty$, $p \equiv \bar{p}/(\rho_\infty v_\infty^2)$, $x \equiv \bar{x}/l_c$, $z \equiv \bar{z}/l_c$, $a \equiv (l_c/\bar{v}_\infty) d\bar{u}/d\bar{x}|_\infty$, $\theta_g \equiv T_g/T_\infty$, $\theta_s \equiv T_s/T_\infty$ and $\kappa \equiv aK/l_c^2$.

The variable changes are given by:

$$u = a\,x\,U(z), \quad v = -a^{1/2} f,$$

$$p_0 - p = \frac{1}{2} Pr\, a^2 \left(1 + \frac{1}{\kappa\Gamma}\right)\left(x^2 + \frac{2F(z)}{a}\right), \quad \eta = a^{1/2} z$$

In the following, the non-dimensional conservation equations are presented with the transformations already performed. The mass conservation equation is given by:

$$U = \frac{df}{d\eta} \tag{1}$$

The equation for the momentum conservation is given by:

$$\frac{Pr}{\Gamma}\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} - \left(\frac{df}{d\eta}\right)^2 - \varepsilon\frac{Pr}{\kappa\Gamma}\frac{df}{d\eta} = -\varepsilon Pr\left(1 + \frac{1}{\kappa\Gamma}\right) \tag{2}$$

It is worth to note the existence of the Darcy term in the above equation, it represents the resistive force due to the existence of the tortuous channels of the porous matrix and the viscous interaction between the flow and the walls of the porous.

We also use equations for the energy, since we consider thermal non-equilibrium for the phases:

$$\frac{\varepsilon}{\Gamma}\frac{d^2\theta_g}{d\eta^2} + \varepsilon f\frac{d\theta_g}{d\eta} = -\frac{N_v}{a}\left(\theta_s - \theta_g\right) \tag{3}$$

$$0 = -(1-\varepsilon)\frac{d^2\theta_s}{d\eta^2} + \frac{N_v}{a}(\theta_s - \theta_g) \tag{4}$$

in which we presented the energy equation for the flowing gas and for the solid matrix, respectively. The parameter a is the non-dimensional strain-rate.

The necessary boundary conditions are given by:

$$f = \frac{df}{d\eta} = \theta_s - \theta_0 = \theta_g - \theta_0 = 0 \;\text{ for }\; \eta = 0$$

$$\frac{df}{d\eta} - 1 = \theta_g - 1 = \theta_s - 1 = 0 \;\text{ for }\; \eta = \infty$$

From the non-dimensionalization process, parameters that controls some properties of the flow emerges. In the above formulation, we obtained:

$$Pr \equiv \nu/\alpha_g, \quad \Gamma \equiv \bar{\lambda}_s/\bar{\lambda}_g$$

in which Pr is the Prandtl number, Γ is a ratio between solid and gas thermal conductivities, and α_g is the gas thermal diffusivity.

The formulation presented here is an extension for the stagnation point flow, with the additional consideration of the presence of a porous medium and the existence of a heat exchange between phases. The heat exchange in the present work will be considered high, of order Γ, and the porosity is low, in such a way that the non-dimensional permeability parameter will be of order $1/\Gamma^2$.

When one analyze the governing equations using the asymptotic expansion method, two characteristic zones emerge. Those zones scales with the solid and gas thermal conductivities, and are denoted in this work as outer zone and inner zone.

2.1 Outer zone

In the outer zone, the flow is Darcyan type and the viscous wall effects are not significantly noted.

Utilizing the assumption of low permeability medium considered before ($1/(\kappa\Gamma) = \beta\Gamma$, where β is a parameter of the order of unity) and performing a expansion given by $f = f_{(0)} + \Gamma^{-1} f_{(1)} + O(\Gamma^{-2})$, we obtain from Eq. (2):

$$f^{'}_{(0)} = 1 \tag{5}$$

and

$$\varepsilon\beta Pr\, f^{'}_{(1)} = \varepsilon Pr + f_{(0)} f^{''}_{(0)} - \left(f^{'}_{(0)}\right)^2 \tag{6}$$

The boundary conditions are:

$$f_{(0)}(0) = f_{(1)}(0) = 0,$$

$$f^{'}_{(0)}\Big|_{\infty} = 1, \quad f^{'}_{(1)}\Big|_{\infty} = U_1$$

in which U_1 is the higher order term from the expansion of the boundary condition for U and will be obtained from Eq. (9).

After solving the above set of equations, we obtain the momentum in the outer zone:

$$f(\eta) = \eta - \Gamma^{-1}\frac{1-\varepsilon Pr}{\varepsilon\beta Pr}\eta + O\left(\Gamma^{-2}\right) \tag{7}$$

and $U_1 = -\,(1-\varepsilon Pr)/(\varepsilon\beta Pr)$.

In the outer zone, gas and solid temperatures are equal due to the high interphase heat exchange. Considering this fact and expanding the temperature as $\theta(\eta) = \theta_{(0)} + \Gamma^{-1}\theta_{(1)} + O(\Gamma^{-2})$, we must solve:

$$\gamma\theta^{''}_{(0)} + f_{(0)}\theta^{'}_{(0)} = 0 \tag{8}$$

$$\gamma\theta^{''}_{(1)} + f_{(0)}\theta^{'}_{(1)} = -\,\theta^{''}_{(0)} - f_{(1)}\theta^{'}_{(0)} \tag{9}$$

in which we defined $\gamma = (1-\varepsilon)/\varepsilon$.

The boundary conditions to be obeyed are given by:

$$\theta_{(0)}(0) = \theta_0, \quad \theta_{(1)}(0) = 0,$$

$$\theta_{(0)}(\eta\to\infty) = 1, \quad \theta_{(1)}(\eta\to\infty) = 0$$

Defining $\Theta = (\theta_0 - 1)$, the temperature profile in the outer zone will given by:

$$\theta(\eta) = \theta_0 - \Theta\, erf\left(\frac{\eta}{\sqrt{2\gamma}}\right) + \Gamma^{-1}\frac{\Theta}{2\gamma}\sqrt{\frac{2}{\pi\gamma}}\left(1+\frac{(1-\varepsilon Pr)}{\varepsilon\beta Pr}\gamma\right)\eta\, exp\left(-\frac{\eta^2}{2\gamma}\right) + O\left(\Gamma^{-2}\right) \tag{10}$$

2.2 Inner zone

In order to capture the processes changes in the inner zone, we must perform a spatial coordinate change given by $\tilde{\eta} = \Gamma\eta$. A decoupling between temperature profiles is observed, due to the different values for the thermal conductivities. However, at the wall both temperatures have the same value, since the gas is in a steady-state.

One must also note that since the gas is in a nearly steady-state, it is necessary to perform a re-scaling for f, as $\tilde{f} = \Gamma f$. Under those considerations, and performing a expansion given by $\tilde{f} = \tilde{f}_{(0)} + \Gamma^{-1}f_{(1)} + O(\Gamma^{-2})$, we must solve the following set of differential equations for the first two terms of the momentum in the inner zone:

$$\tilde{f}^{'''}_{i(0)} - \varepsilon\beta\left(\tilde{f}^{'}_{i(0)} - 1\right) = 0 \tag{11}$$

$$Pr\tilde{f}^{'''}_{(1)} - \varepsilon\beta Pr\tilde{f}^{'}_{(1)} + \tilde{f}_{(0)}\tilde{f}^{''}_{(0)} - \left(\tilde{f}^{'}_{(0)}\right)^2 = -\varepsilon Pr \tag{12}$$

The following boundary conditions must be obeyed:

$$\tilde{f}_{(0)}(0) = \tilde{f}^{'}_{(0)}(0) = 0,$$

$$\tilde{f}_{(1)}(0) = \tilde{f}^{'}_{(1)}(0) = 0,$$

$$\tilde{f}^{'}_{(0)}\Big|_{\tilde{\eta}\to\infty} = f^{'}_{(0)}\Big|_{\eta\to 0} = 1,$$

$$\tilde{f}^{'}_{(1)}\Big|_{\tilde{\eta}\to\infty} = f^{'}_{(1)}\Big|_{\eta\to 0} = -\,\frac{(1-\varepsilon Pr)}{\varepsilon\beta Pr}$$

Note that the fluxes in the inner zone must match with the fluxes in the outer zone. For this system, we obtain the momentum expression for the inner zone:

$$\tilde{f}(\tilde{\eta}) = \tilde{\eta} + \frac{1}{\sqrt{\varepsilon\beta}}\left(e^{-\sqrt{\varepsilon\beta}\tilde{\eta}} - 1\right) - \Gamma^{-1}\left[\frac{(1-\varepsilon Pr)}{\varepsilon\beta Pr}\tilde{\eta} + e^{-\sqrt{\varepsilon\beta}\tilde{\eta}}\frac{\left(10\tilde{\eta} + 2\sqrt{\varepsilon\beta}\tilde{\eta}^2\right)}{8\varepsilon\beta Pr}\right. +$$

$$\left.\frac{\left(e^{-\sqrt{\varepsilon\beta}\tilde{\eta}}-1\right)}{8\varepsilon\beta Pr}\frac{\left(10+8\left(1-\varepsilon Pr\right)\right)}{\sqrt{\varepsilon\beta}}\right]+O\left(\Gamma^{-2}\right) \tag{13}$$

In the inner zone, the temperature profiles decouple due to the difference in both thermal conductivities. For the energy analysis, we consider a intense interphase heat exchange, and quantify such exchange by $N_v = \Gamma\, n_v$, where n_v is a parameter of the order of unity. We consider a heated wall at a constant temperature θ_0.

Both temperatures are expanded, in a general form, as $\theta = \theta_{(0)} + \Gamma^{-1}\theta_{(1)} + O(\Gamma^{-2})$. Under such considerations, we must solve the following set of differential equations for the first terms of the temperature in the inner zone:

$$\varepsilon\theta^{''}_{g(0)} = -n_v\left(\theta_{s(0)} - \theta_{g(0)}\right), \tag{14}$$

$$\varepsilon\theta^{''}_{g(1)} + \varepsilon\tilde{f}_{i(0)}\theta^{'}_{g(0)} = -n_v\left(\theta_{s(1)} - \theta_{g(1)}\right) \tag{15}$$

$$(1-\varepsilon)\theta^{''}_{s(0)} = 0, \tag{16}$$

$$(1-\varepsilon)\theta^{''}_{s(1)} = n_v\left(\theta_{s(0)} - \theta_{g(0)}\right) \tag{17}$$

Those equations are subjected to the following boundary and matching conditions:

$$\theta_{g(0)}(0) = \theta_{s(0)}(0) = \theta_0,$$

$$\theta_{g(1)}(0) = \theta_{s(1)}(0) = 0$$

$$\theta^{'}_{g(0)}\Big|_{\tilde{\eta}\to\infty} = \theta^{'}_{s(0)}\Big|_{\tilde{\eta}\to\infty} = \theta^{'}_{(0)}\Big|_{\eta\to 0} = -\,\Theta\sqrt{\frac{2}{\pi\gamma}}$$

$$\theta^{'}_{g(1)}\Big|_{\tilde{\eta}\to\infty} = \theta^{'}_{s(1)}\Big|_{\tilde{\eta}\to\infty} = \theta^{'}_{(1)}\Big|_{\eta\to 0} = \frac{\Theta}{2\gamma}\sqrt{\frac{2}{\pi\gamma}}\left(1+\frac{(1-\varepsilon Pr)}{\varepsilon\beta Pr}\gamma\right)$$

For the solid phase, we obtain a solution given by:

$$\theta_s(\tilde{\eta}) = \theta_0 - \Theta\sqrt{\frac{2}{\pi\gamma}}\tilde{\eta} \;+\; \Gamma^{-1}\frac{\Theta}{2\gamma}\sqrt{\frac{2}{\pi\gamma}}\left(1+\frac{(1-\varepsilon Pr)}{\varepsilon\beta Pr}\gamma\right)\tilde{\eta} + O\left(\Gamma^{-2}\right) \tag{18}$$

For the gas phase, we recall that we are considering a low porosity medium, so $1/\varepsilon^2 \gg 1$, with this in mind, we solve a simplified form of the Eq. (15), obtaining a result for the gas phase temperature profile given by:

$$\theta_g(\tilde{\eta}) = \theta_0 - \Theta\sqrt{\frac{2}{\pi\gamma}}\tilde{\eta} + \Gamma^{-1}\left[\Theta\sqrt{\frac{2\varepsilon}{\pi\gamma\beta}}\frac{1}{n_v} + \sqrt{\frac{2}{\pi\gamma}}\frac{\Theta}{2\gamma}\left(1+\frac{(1-\varepsilon Pr)}{\varepsilon\beta Pr}\gamma\right)\tilde{\eta} + \right.$$

$$\left. +\,\Theta\sqrt{\frac{2}{\pi\gamma\varepsilon\beta}}\left(\frac{\varepsilon}{\varepsilon^2\beta - n_v}\right)exp\left(-\sqrt{\varepsilon\beta}\tilde{\eta}\right) - \Theta\sqrt{\frac{2\varepsilon\beta}{\pi\gamma}}\frac{\varepsilon^2}{n_v\left(\varepsilon^2\beta - n_v\right)}exp\left(-\sqrt{\frac{n_v}{\varepsilon}}\tilde{\eta}\right)\right] + O\left(\Gamma^{-2}\right) \tag{19}$$

Those analytical results will be shown in a graphical form and their results will be analyzed in the next section.

3. RESULTS AND DISCUSSION

The inner zone, of length of order $1/\Gamma$, corresponds to the boundary layer, thermal and viscous, near the wall. When one analyze the problem from the outer zone, those boundary layer effects are lost, and one can only note them in the inner zone, that is the reason why it is necessary to re-scale the spatial coordinate near the wall.

Such problem is similar to the problem of the boundary layer created in the flow parallel to a flat plate. In this classical problem, the majority of the velocity field is view as a uniform profile. However, right on the flat plate the fluid velocity is zero, due to the non-slip condition arising from the viscous attachment. So, one note the viscous effect only in a small region in the vicinity of the plate. Analyzing Fig. (1) we observe that in the outer zone the viscous dissipative effect is lost.

From the inner zone analysis, we obtained the boundary layer effect in the velocity field at the vicinity of the wall, as one can see from Fig. (2).

From the thermal analysis, the temperature of the solid matrix and of the gas in the outer zone are at equilibrium. Both of them are affected by the presence of the heated wall, and as the flow approaches the stagnation-point, they sense a exponential increase in their temperature. Temperature profiles in the outer zone considering different wall temperatures and for the inner zone considering the wall temperature to be $\theta_0 = 2$ are shown in Fig. (3).

When we analyze the inner zone, we note that the solid and the gaseous profiles matches almost exactly, even though their expressions are different, as exhibited by Eqs. (18) and (19). This is a consequence of the fact that, although they are different valued, their leading order terms are equal. Such may be explained by the fact that in the inner zone we are very close to the wall, and the velocity field is nearly zero. Due to the gas being almost at a zero velocity, the gas and the solid matrix reach an equilibrium temperature very quickly, differing only in small order correction terms, as one can see from Fig. (4).

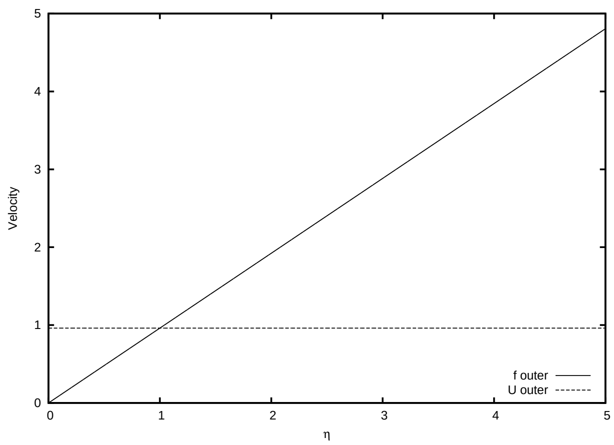

Figure 1. Velocity in the outer zone

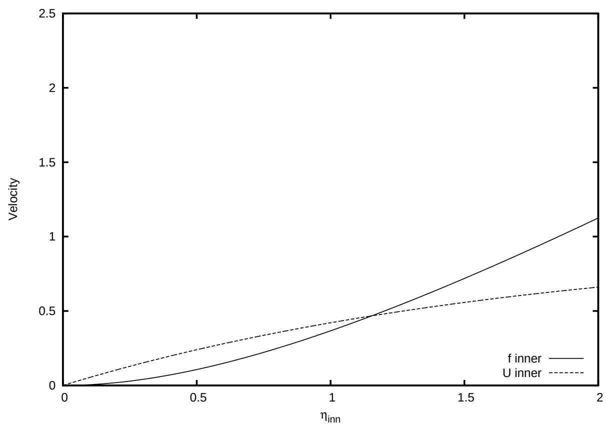

Figure 2. Velocity in the inner zone

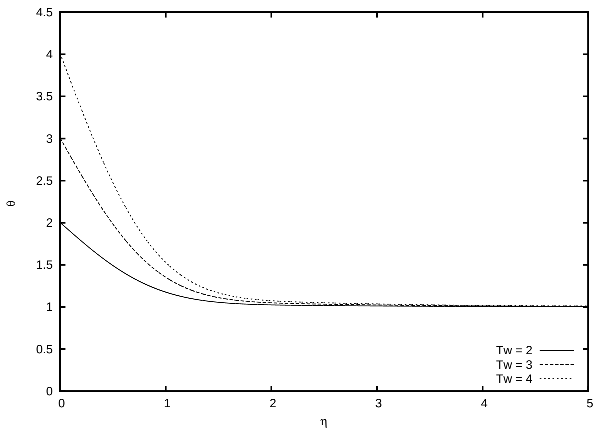

Figure 3. Temperature in the outer zone

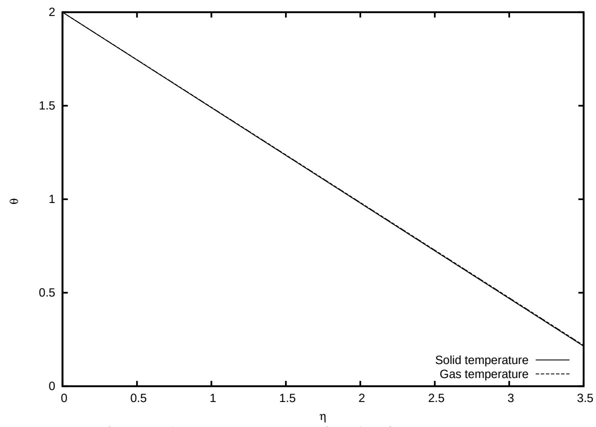

Figure 4. Temperature in the inner zone

4. CONCLUSIONS

The Hiemenz problem inside a porous medium with the consideration of the existence of a constant wall temperature was analyzed analytically. A low porosity medium was considered and asymptotic expansions were applied in order to solve the thermal-momentum coupled problem.

Two characteristic zones emerged from the analysis: outer zone and inner zone. The first one scales with the solid thermal conductivity and its governing equation for the momentum is a modified Darcy equation. The second one scales with the gas thermal conductivity and it contemplates the viscous effects near the wall.

The high value for the interphase heat exchange is responsible to bring the solid and the gas temperature into equilibrium far from the wall. When the flow is near the wall the phases are not in equilibrium, and although their leading order terms are equal they differ in higher order terms. This small decoupling comes from the fact that in the inner zone the difference in thermal conductivities becomes relevant in the process of heat transfer. However, at a small distance from the wall the gas have a very low velocity, so that there is enough time to solid and gas reach almost to an equilibrium temperature. That is why the profiles matches only in leading order terms.

Numerical analysis of the proposed problem exist in the literature, but an analytical approach was not found by the authors. At proper limits, the modeling proposed re-obtain a set of previous problems (Attia, 2007; Wu et al., 2005), validating the approach, since the results matches.

In future works, a constant wall heat flux is to be considered, instead of a constant wall temperature.

5. ACKNOWLEDGEMENTS

This work was in part supported by Master in Science Program from Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - CAPES.

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