Celest Mech Dyn Astr (2009) 104:337–353
DOI 10.1007/s10569-009-9216-3

ORIGINAL ARTICLE

Attitude stability of artificial satellites subject to gravity gradient torque

Rodolpho Vilhena de Moraes · Regina Elaine Santos Cabette · Maria Cecília Zanardi · Teresinha J. Stuchi · Jorge Kennety Formiga

Received: 19 December 2007 / Revised: 25 February 2009 / Accepted: 29 May 2009 / Published online: 5 July 2009

Abstract The stability of the rotational motion of artificial satellites is analyzed considering perturbations due to the gravity gradient torque, using a canonical formulation, and Andoyer's variables to describe the rotational motion. The stability criteria employed requires the reduction of the Hamiltonian to a normal form around the stable equilibrium points. These points are determined through a numerical study of the Hamilton's equations of motion and linear study of their stability. Subsequently a canonical linear transformation is used to diagonalize the matrix associated to the linear part of the system resulting in a normalized quadratic Hamiltonian. A semi-analytic process of normalization based on Lie–Hori algorithm is applied to obtain the Hamiltonian normalized up to the fourth order. Lyapunov stability of the equilibrium point is performed using Kovalev and Savchenko's theorem. This semi-analytical approach was applied considering some data sets of hypothetical satellites, and only a few cases of stable motion were observed. This work can directly be useful for the satellite maintenance under the attitude stability requirements scenario.

R. V. de Moraes (✉) · R. E. S. Cabette · M. C. Zanardi
Universidade Estadual Paulista Júlio de Mesquita Filho, Campus de Guaratinguetá, Guaratinguetá, SP 12516-410, Brazil
e-mail: rodolpho@feg.unesp.br

R. E. S. Cabette · J. K. Formiga
Instituto Nacional de Pesquisas Espaciais, São José dos Campos, SP 12227-010, Brazil

R. E. S. Cabette
e-mail: recabette@uol.com.br

T. J. Stuchi
Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ 21 941-909, Brazil
e-mail: tstuchi@if.ufrj.br

M. C. Zanardi
e-mail: cecilia@feg.unesp.br

J. K. Formiga
e-mail: jkennety@dem.inpe.br

Keywords Rotational motion of artificial satellites · Non linear stability · Normalization

1 Introduction

The study of the attitude of artificial satellites is usually divided into control, determination and propagation. The attitude propagation is associated with the prediction of the attitude at each instant. Attitude control is the process to command a desired attitude while the attitude determination is the ability to know the spacecraft attitude using information acquired by instruments on board of the spacecraft. In controlling a spacecraft the analysis of the stability is almost mandatory. The stability analysis of the rotational motion of spacecrafts taking into account external torques is a very important issue in the attitude maintenance to ensure the success of the spatial missions. Conditions for the existence of equilibrium points and general form for stability conditions have been studied recently for the attitude motion of artificial satellites subjected to external torques (Sarychev et al. 2007, 2008). See also Celletti and Sidorenko (2008).

The gravity gradient torque derivation in the current analysis makes use of canonical transformations (de Moraes 1989).

The rotational motion of artificial satellites is described through Eulerian angles and the components of the rotation speed vector in the satellite's principal system of inertia. In view of the canonical treatment of the problem, canonical Andoyer variables, related with the Eulerian angles are used (Kinoshita 1972; Zanardi 1986).

Andoyer's canonical variables (Kinoshita 1972) are represented by (L_1, L_2, L_3) and (l_1, l_2, l_3) as described in Fig. 1. The angular variables l_1, l_2, l_3 are angles related to the satellite system Oxyz (with axes parallel to the system of the spacecraft's principal axes of inertia) and the equatorial system OXYZ (with axes parallel to the axes of the Earth equatorial system). Andoyer's metric variables L_1, L_2, L_3 are defined as: L_2 is the modulus of the rotational angular momentum vector $\vec{L}_2$, L_1, and L_3 are, respectively, the projection of $\vec{L}_2$ on the z-axis' principal axis system of inertia ($L_1 = L_2 \cos J_2$, where J_2 is the angle between z-satellite axis and $\vec{L}_2$) and on the Z-equatorial axis ($L_3 = L_2 \cos II_2$, where II_2 is the angle between Z-equatorial axis and $\vec{L}_2$).

The objective of the Hamiltonian, taken here to a fourth order normal form, is to verify whether linearly stable equilibrium points remain stable under the influence of higher orders,

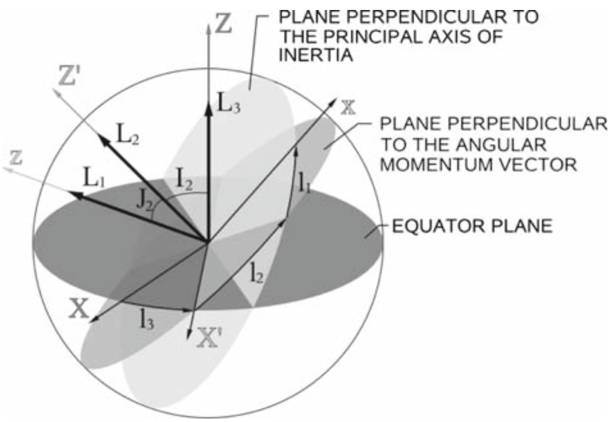

Fig. 1 Andoyer variables ($L_1, L_2, L_3, l_1, l_2, l_3$)

essentially up to fourth order perturbations according to the criteria established by Kovalev–Savchenko theorem (Kovalev and Savchenko 1975). The use of normal forms to analyze the stability of equilibrium points has been considered for several Hamiltonian dynamical systems of Space Mechanics (Kovalev and Savchenko 1975; Chudnenko 1981; Elipe and Ferrer 1985; Elipe and Lopez-Moratalla 2006; Mansilla 2006).

In this work simulations were done for two groups of hypothetical artificial satellites, classified here as medium or small sized satellites according to their physical and geometrical characteristics. The non-linear stability of the equilibrium points of these satellites is analyzed using the three necessary conditions of the theorem of Kovalev and Savchenko (Kovalev and Savchenko 1975).

2 Equations of motion

In this section, the Hamiltonian of the rotational motion of an artificial satellite in a Keplerian translational motion and under the influence of the gravity torque is introduced. The Andoyer variables, as defined in the previous section, are used to characterize the motion of the satellite around its center of mass and the Delaunay variables to describe the motion of the center of mass of the satellite around the Earth (Kinoshita 1972). The Delaunay variables (L, G, H, l, g, h) are defined as: $L = \sqrt{\mu a}$, $G = L\sqrt{1-e^2}$, $H = G\cos i$, l is the mean anomaly, g is the argument of the perigee, h is the longitude of the ascending node, μ is the gravitational parameter of the earth, i is the orbital inclination, a and e are respectively the orbital semi-major axis and eccentricity of the satellite.

In this paper it is assumed that the orbit is known and given by the Keplerian motion. The expansion of the Hamiltonian associated with the gravity gradient torque is truncated in the first order of the eccentricity. This simplification was adopted in order to simplify some analytical and computational efforts.

The Hamiltonian of the considered problem, expressed in terms of the Andoyer and Delaunay variables (Zanardi 1986) can be written as:

$$\begin{aligned} &F(L_1, L_2, L_3, l_2, l_3, L, G, H, l, g, h) \\ &\quad = F_0(L, L_1, L_2) + F_1(L_1, L_2, L_3, l_2, l_3, L, G, H, l, g, h) \end{aligned} \tag{1}$$

including the gravity gradient torque we have:

$$\begin{aligned} F_0 = &-\frac{\mu^2 M^3}{2L^2} + \frac{1}{2}\left(\left(\frac{1}{C} - \frac{1}{2A} - \frac{1}{2B}\right) L_1^2 + \frac{1}{2}\left(\frac{1}{A} + \frac{1}{B}\right) L_2^2\right) \\ &+ \frac{1}{4}\left(\frac{1}{B} - \frac{1}{A}\right)(L_2^2 - L_1^2)\cos 2l_1 \end{aligned} \tag{2}$$

$$F_1 = \left(\frac{\mu^4 M^6}{L^6}\right)\left[\frac{2C - A - B}{2} h_1(l_m, L_n) + \frac{A - B}{4} h_2(l_m, L_n)\right] \tag{3}$$

with: $m = 2, 3$ and $n = 1, 2, 3$; A, B and C being the satellite's principal moments of inertia; h_1 and h_2 are functions of the marked variables, where l_2 and l_3 appear in the arguments of cosines. The complete analytical expression for F_1 is given in "Appendix".

The equations of motion associated to the Hamiltonian function F are given by:

$$\begin{cases} \frac{\mathrm{d}l_i}{\mathrm{d}t} = \frac{\partial F}{\partial L_i}, \\ \frac{\mathrm{d}L_i}{\mathrm{d}t} = -\frac{\partial F}{\partial l_i}; \end{cases} \quad (i = 1, 2, 3). \tag{4}$$

The equations (4) are used to determine the equilibrium points of the rotational motion in the Sect. 5 of this paper. In the current application the satellites will be considered nearly symmetrical, that is $A \approx B$, and in this case the Hamiltonian do not depend upon the variable l_1.

3 Normal form for the Hamiltonian

The normalization of the Hamiltonian system in the neighborhood of a linearly stable position of equilibrium will be obtained up to fourth order terms so that we can apply the stability criteria mentioned above. In order to do this, first it is necessary to diagonalize the quadratic terms of the Hamiltonian.

Since the variable l_1 is cyclic, the Hamiltonian is of the form $F(L_2, L_3, l_2, l_3)$ and the Eq. 4 can be written as:

$$\dot{w} = J\, F_w \tag{5}$$

where w is the state vector and F_w is the matrix of the derivatives with respect the considered variables and given by:

$$w = \begin{bmatrix} l_2 \\ L_2 \\ l_3 \\ L_3 \end{bmatrix}, \quad F_w = \begin{bmatrix} F_{l_2} \\ F_{L_2} \\ F_{l_3} \\ F_{L_3} \end{bmatrix} \tag{6}$$

and

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{7}$$

is a symplectic matrix.

Let us consider the Taylor expansion of the Hamiltonian F in the neighborhood of the equilibrium, viz.:

$$F(l_m, L_n) = \sum_{k=2}^{\infty} F_k(l_m, L_n) = F_2(l_m, L_n) + F_3(l_m, L_n) + F_4(l_m, L_n) + \cdots \tag{8}$$

where $n = 1, 2, 3$, $m = 2, 3$, and F_k, $k = 2, 3, 4, \ldots$, are terms of order k.

The Hessian matrix is given by:

$$\mathbf{P} = \begin{pmatrix} \frac{\partial^2 F}{\partial l_2^2} = f_{11} & \frac{\partial^2 F}{\partial l_2 \partial L_2} = f_{12} & \frac{\partial^2 F}{\partial l_2 \partial l_3} = f_{13} & \frac{\partial^2 F}{\partial l_2 \partial L_3} = f_{14} \\ \frac{\partial^2 F}{\partial L_2 \partial l_2} = f_{21} & \frac{\partial^2 F}{\partial L_2^2} = f_{22} & \frac{\partial^2 F}{\partial L_2 \partial l_3} = f_{23} & \frac{\partial^2 F}{\partial L_2 \partial L_3} = f_{24} \\ \frac{\partial^2 F}{\partial l_3 \partial l_2} = f_{31} & \frac{\partial^2 F}{\partial l_3 \partial L_2} = f_{32} & \frac{\partial^2 F}{\partial l_3^2} = f_{33} & \frac{\partial^2 F}{\partial l_3^{\partial} L_3} = f_{34} \\ \frac{\partial^2 F}{\partial L_3 \partial l_2} = f_{41} & \frac{\partial^2 F}{\partial L_3 \partial L_2} = f_{42} & \frac{\partial^2 F}{\partial L_3 \partial l_3} = f_{43} & \frac{\partial^2 F}{\partial L_3^2} = f_{44} \end{pmatrix} \tag{9}$$

where the second derivatives are evaluated at the equilibrium point around which the expansion is performed.

If the eigenvalues of the matrix $\mathbf{J}P$ are simple and ordered, then $\mathbf{J}P$ can be diagonalized. This process is straightforward but laborious so it was calculated with the software

MAPLE. The automatic normalization of the eigenvectors given by MAPLE does not ensure a canonical transformation (Stuchi 2002), hence corrections need to be applied.

Considering the linear canonical change:

$$w = Dz, \tag{10}$$

with

$$z = (q_1, p_1, q_2, p_2) \tag{11}$$

and

$$D = \Gamma N \tag{12}$$

The matrix Γ has as its four columns the components of the eigenvectors (obtained with MAPLE) associated to the eigenvalues of the matrix $\mathbf{J}P$. D is a symplectic matrix obtained with an auxiliary matrix R given by (Stuchi 2002):

$$R = \Gamma^T \mathbf{J} \Gamma = \begin{pmatrix} 0 & r_1 & 0 & 0 \\ -r_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2 \\ 0 & 0 & -r_2 & 0 \end{pmatrix} \tag{13}$$

and N is given by

$$N = \begin{pmatrix} 1/\sqrt{r_1} & 0 & 0 & 0 \\ 0 & 1/\sqrt{r_1} & 0 & 0 \\ 0 & 0 & 1/\sqrt{r_2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{r_2} \end{pmatrix}. \tag{14}$$

Therefore, the transformation is now canonical since:

$$D^T \mathbf{J} D = J \tag{15}$$

Up to the second order the system of equations in terms of the variable $\mathbf{z} = (q_1, p_1, q_2, p_2)$ is given by:

$$\dot{z} = \{H_2, z\} \tag{16}$$

where H_2 in the normal form expressed as:

$$H_2\,(q_i, p_i) = \sum_{i=1}^{2} \upsilon_i \left(q_i^2 + p_i^2\right), \tag{17}$$

with $\upsilon_i = \frac{\lambda_i}{2}$, where $\boldsymbol{\lambda}_i$ $(i = 1, 2)$ is the ith purely imaginary eigenvalue of $\mathbf{J}P$.

In other words, the second order Hamiltonian (17) is in the canonical form around a center–center equilibrium point.

According to Stuchi (Stuchi 2002) it is convenient to introduce a complex change of coordinates,

$$\begin{pmatrix} q_i \\ pi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \qquad i = 1, 2 \tag{18}$$

with the inverse

$$\begin{pmatrix} x_i \\ yi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix} \begin{pmatrix} q_i \\ p_i \end{pmatrix}, \qquad i = 1, 2 \tag{19}$$

to facilitate the normalization procedure.

Thus, the quadratic part of the Hamiltonian in complex normal form is represented by HC_2 and given by:

$$\mathrm{HC}_2 = H_2(x_i, y_i) = \sum_{i=1}^{2} \lambda_i \quad x_i y_i. \tag{20}$$

as a result of the transformation D, followed by the complexification.

With the Hamiltonian written in terms of the complex variables and the quadratic term HC_2 already in normal form by applying the Lie–Hori method (Hori 1966) to take the higher order terms $H_3, H_4, \ldots, H_n$ to normal form up to the required order. A normalization up to fourth order is sufficient (Kovalev and Savchenko 1975). An expansion up to fourth order can be done using the software MAPLE. If higher order is desired then an automatic procedure is recommended for the normalization using a code constructed specially for this purpose (Stuchi 2002; Machuy 2001).

4 Hori–Lie series method

The normal form is obtained from the expansion of the Hamiltonian in terms of the Lie series given by (Hori 1966; Ferraz-Mello 2007):

$$H^{\text{new}} = H + \left\{H, \overline{\overline{G}}\right\} + \frac{1}{2!}\left\{\left\{H, \overline{\overline{G}}\right\}, \overline{\overline{G}}\right\} + \frac{1}{3!}\left\{\left\{\left\{H, \overline{\overline{G}}\right\}, \overline{\overline{G}}\right\}, \overline{\overline{G}}\right\} + \cdots \tag{21}$$

where $\overline{\overline{G}} = \overline{\overline{G}}(x, y)$ is the generating function, $H = H(x, y)$ is the original Hamiltonian and $\left\{H, \overline{\overline{G}}\right\} = \frac{\partial H}{\partial x}\frac{\partial \overline{\overline{G}}}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial \overline{\overline{G}}}{\partial x}$ is the usual Poisson bracket. We recall that the Poisson bracket of two homogeneous polynomials of degrees r and s, respectively, is a homogeneous polynomial of degree $r + s - 2$ and this fact is very useful in our calculations.

The Hamiltonian H^{new} is the result of the application of a near identity canonical transformation $T_{\overline{\overline{G}}}$ given by the generating function $\overline{\overline{G}}(x, y)$ in $H(x, y)$ through $(x, y) = T_{\overline{\overline{G}}}(q, p)$ so that:

$$\begin{aligned} H^{\text{new}}(x, y) &= H(T^{-1}_{\overline{\overline{G}}}(x, y)) \\ H^{\text{new}} &= H^{\text{new}}_2 + H^{\text{new}}_3 + H^{\text{new}}_4 + \cdots \\ H^{\text{new}} &= \lambda_1 \Im_1 + \lambda_2 \Im_2 + \sum_{m_1+m_2>2} \Im_1^{m_1} \Im_2^{m_2} \end{aligned} \tag{22}$$

where $\Im_k = \tilde{x}_k \tilde{y}_k$.

The new Hamiltonian, ordered by degree up to fourth order, has the following form (Stuchi 2002; Ferraz-Mello 2007):

$$H^{\text{new}}_2 = H_2, \tag{23}$$

$$H^{\text{new}}_3 = H_3 + \left\{H_2, \overline{\overline{G}}_3\right\}, \tag{24}$$

$$H^{\text{new}}_4 = H_4 + \left\{H_3, \overline{\overline{G}}_3\right\} + \frac{1}{2!}\left\{\left\{H_2, \overline{\overline{G}}_3\right\}, \overline{\overline{G}}_3\right\} + \left\{H_2, \overline{\overline{G}}_4\right\} \tag{25}$$

Since H_2^{new} has already been put in normal form through the diagonalization procedure. Following steps eliminate all monomials of H_3^{new} and some of H_4^{new}:

I: As mentioned in Eq. 22, the dependency of the normal form Hamiltonian is usually expressed in powers of $\boldsymbol{\Im_k = \tilde{x}_k \tilde{y}_k}$, the action variables, is given by:

$$H^{\text{new}} = \lambda_1 \Im_1 + \lambda_2 \Im_2 + \sum_{m_1+m_2>2} \Im_1^{m_1} \Im_2^{m_2}, \tag{26}$$

so that we can easily see that:

$$H_3^{\text{new}} = 0. \tag{27}$$

Therefore, it is necessary to find $\overline{\overline{\mathbf{G}}}_3$ so that H_3^{new} is zero. The third degree generating function $\overline{\overline{\mathbf{G}}}_3$ is determined from the third order homological equation:

$$\left\{H_2, \overline{\overline{G}}_3\right\} = -H_3, \tag{28}$$

$$\begin{aligned} \sum \frac{\partial H_2}{\partial x}\frac{\partial \overline{\overline{G}}_3}{\partial y} - \frac{\partial H_2}{\partial y}\frac{\partial \overline{\overline{G}}_3}{\partial x} &= -\sum h_{j_x,j_y} x^{j_x} y^{j_y} \\ \sum \lambda y\, j_y\, g_{j_x,j_y}\, x^{j_x} y^{j_y-1} - \lambda x\, j_x g_{j_x,j_y} x^{j_x-1} y^{j_y} &= -\sum h_{j_x,j_y} x^{j_x} y^{j_y} \\ \sum \lambda (j_y - j_x) g_{j_x,j_y} x^{j_x} y^{j_y-1} &= -\sum h_{j_x,j_y} x^{j_x} y^{j_y} \end{aligned} \tag{29}$$

where

$$g_{j_x,j_y} = \frac{-h_{j_x j_y}}{\sum \lambda (j_y - j_x)} \tag{30}$$

with the non resonance condition $\sum \lambda (j_y - j_x) \neq 0$, which is of course satisfied by the monomials of order three.

II: A similar process is repeated for H_4^{new} and $\overline{\overline{G}}_4$ computing the Poisson brackets of the Eq. 25. The terms that remain in $H_4^{\text{new}}(\tilde{x}_k, \tilde{y}_k)$, $(\tilde{x}_k, \tilde{y}_k)$ being the normal complex variables, are denominated resonant monomials (Machuy 2001). Equation 25 is deduced in the following form:

$$\begin{aligned} \left\{\overline{\overline{\mathbf{G}}}_4, \mathbf{H}_2\right\} + \mathbf{H}_4^{\mathbf{new}} &= \mathbf{H_4} + \left\{\mathbf{H_3}, \overline{\overline{\mathbf{G}}}_\mathbf{3}\right\} + \tfrac{\mathbf{1}}{\mathbf{2}} \left\{\left\{\mathbf{H_2}, \overline{\overline{\mathbf{G}}}_\mathbf{3}\right\}, \overline{\overline{\mathbf{G}}}_3\right\} \\ \left\{\overline{\overline{\mathbf{G}}}_4, \mathbf{H}_2\right\} + \mathbf{H_4^{new}} &= \mathbf{F_4} \end{aligned} \tag{31}$$

The last equation is the homological equation of fourth order. The resonant terms, i.e., the ones which cannot be eliminated, are grouped as

$$R_4 = F_4 - H_4^{\text{new}} \tag{32}$$

so that the terms of fourth order in the generating function are:

$$\overline{\overline{G}}_{j_x j_y} = \frac{r_{j_x j_y}}{\sum \lambda (j_x - j_y)} \tag{33}$$

with $\sum \lambda (j_y - j_x) \neq 0$, otherwise they constitute the fourth order terms of the normal form.

III: For higher orders (>4) the procedure is the same but the construction of a specialized code is recommended for the normalization as already mentioned.

5 Computational algorithm for normal form of the Hamiltonian of the rotational motion

A semi-numeric algorithm to compute the normal form of a Hamiltonian system is presented in four steps. This procedure, developed with the MAPLE software, will be applied to the Hamiltonian associated with the rotational motion of an artificial satellite under the influence of the gravity gradient torque, given by Eq. 1.

Step I

The first step is the computation of the equilibrium points of the given dynamical system taking as parameters the initial conditions for the rotational and translational motion and the geometrical and physical characteristics of the satellite in study. These equilibrium points are numerically obtained annulling the gradient of the Hamiltonian F in the right side of the Eq. 4. In order to calculate the equilibrium points from Eq. 4, a procedure was developed using the software MATHEMATICA for each satellite. This procedure involves the derivations of Eq. 4, the substitution of the numerical values of the satellite's parameters and finally the numerical resolution of the resulting algebraic system of equations. Several equilibrium points are obtained for each case under consideration. However, we note that it is not possible to guarantee that all equilibrium points were found.

Step II

For each case and for each equilibrium point the Hamiltonian is expanded in the neighborhood of the equilibrium point and the Hessian is evaluated through Eq. 9.

Subsequently the Hessian is multiplied by the simplectic matrix **J** to get the matrix **J**P to be diagonalized. The eigenvalues and eigenvectors of the matrix JP are found to construct the matrix Γ that diagonalizes **J**P with the default MAPLE normalization. The simplectic matrix D, given by Eq. 12, is then obtained.

The resulting Hamiltonian is:

$$H_2 = \frac{\lambda_1}{2}\left(q_1^2 + p_1^2\right) + \frac{\lambda_2}{2}\left(q_2^2 + p_2^2\right) \tag{34}$$

Introducing complex variables x, p_x, y, and p_y:

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix}\begin{pmatrix} x \\ p_x \end{pmatrix}, \begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix}\begin{pmatrix} y \\ p_y \end{pmatrix} \tag{35}$$

the quadratic part of the Hamiltonian is expressed in complex variables as:

$$\begin{aligned} H_2^{\text{new}} = {} & h_{xp_y} x p_y + h_{p_x p_y} p_x p_y + h_{xy} x\, y + h_{p_{xy}} p_x\, y \\ & + h_y y^2 + h_{p_x} p_x^2 + h_x x^2 + h_{p_y} p_y^2 + \lambda_k y p_y + \lambda_k x p_x \end{aligned} \tag{36}$$

Since the coefficients h_{xy}, h_x, h_y, $h_{p_x} p_y$, h_{xp_y}, h_{p_x}, h_{p_y}, $h_{p_{xy}}$ are null within the required precision we get

$$H_2^{\text{new}} = i\lambda_1(x\, p_x) + i\,\lambda_2(y\, p_y). \tag{37}$$

Step III

Extending the complexification procedure to the terms of higher orders, we get H_3, and the generating function $\overline{\overline{G}}_3$ necessary to make $H_3^{\text{new}} = 0$. The third order Hamiltonian H_3 has 20 monomials and consequently $\overline{\overline{G}}_3$ has same number of monomials. These 20 coefficients are determined from the third order homological Eq. 28 and the generating function $\overline{\overline{G}}_3$ assumes the form:

$$\overline{\overline{G}}_3 = \frac{1}{12}\Bigg\{120 + \sqrt{2} \times \left[\frac{x^3 + (12+3i)x^2 p_x - (3-12i)x\, p_x^2 - i p_x^3 + y\, p_y\,(12+12i)\,(x - p_x)}{\omega_1} + (3+3i)\left[\frac{-y^2 p_x + x\, p_y^2}{\omega_1 - 2\omega_2} + \frac{4\left(y - p_y^2\right)\left(x\, p_x + y\, p_y^2\right)}{\omega_2} + \frac{\left(y\, p_x^2 - x^2 p_y\right)}{\omega_2 - 2\omega_1} + \frac{\left(x^2 y - p_y p_x^2\right)}{\omega_2 + 2\omega_1} + \frac{\left(x\, y^2 - p_y^2 p_x\right)}{\omega_2 + 2\omega_1}\right]\right]\Bigg\} \tag{38}$$

where ω_1 and ω_2 are the linear frequencies, so that $\lambda_1 = i\omega_1$ and $\lambda_2 = i\omega_2$.

The same procedure described in **step II** is used for the fourth order terms leading to H_4 in normal form and the respective generating function $\overline{\overline{G}}_4$. The Hamiltonian H_4 has 35 monomials and some of these monomials cannot be eliminated and can be expressed in terms of the variables $\Im_1$ and $\Im_2$.

Step IV

The last step consists of expressing the Hamiltonian in terms of the real normal variables $\tilde{p}_i$ *e* $\tilde{q}_i$, through the inverse transformation (18), with H_2^{new} and H_4^{new} given by:

$$H_2^{\text{new}} = \upsilon_1\left(\tilde{q}_1^2 + \tilde{p}_1^2\right) + \upsilon_2\left(\tilde{q}_2^2 + \tilde{p}_2^2\right) \tag{39}$$

$$\begin{aligned} H_4^{\text{new}} &= \delta_{11}\left(\tilde{q}_1^4 + \tilde{p}_1^4 + 2\tilde{q}_1^2\tilde{p}_1^2\right) + \delta_{12}\left(\tilde{q}_1^2\tilde{q}_2^2 + \tilde{q}_1^2\tilde{p}_2^2 + \tilde{p}_1^2\tilde{q}_2^2 + \tilde{p}_1^2\tilde{p}_2^2\right) \\ &\quad + \delta_{22}\left(\tilde{q}_2^4 + \tilde{p}_2^4 + 2\tilde{q}_2^2\tilde{p}_2^2\right) \end{aligned} \tag{40}$$

where $\upsilon_i = \frac{\lambda_i}{2}$, $i = 1, 2$, δ_{ij} are combinations of the eigenvalues (λ_1, λ_2), and numerically computed coefficients from Eqs. 24 and 25.

Normal form for the Hamiltonian corresponding to second and fourth orders are given respectively by the Eqs. 39 and 40, is then used in the next section to apply the Kovalev and Savchenko theorem (Kovalev and Savchenko 1975) to analyze the stability of the rotational motion of artificial satellites under the influence of the gravity gradient torque.

6 Stability analysis

Section 7 deals with the analysis of stability of the rotational motion based on Kovalev and Savchenko theorem. Therefore, a brief synopsis of the theorem is presented below.

Let the Hamiltonian $\bar{H}$ be an analytic function of coordinates and impulses at the point $\bar{P}$ corresponding to a steady motion, and let the Hamiltonian $\bar{H}^o$ of the reduced system (normal form) satisfy the following conditions at this point:

1: The eigenvalues of the linear reduced system are pure-imaginary and are $\pm i\alpha_1$ and $\pm i\alpha_2$;
2: The condition $k_1\alpha_1^o + k_2\alpha_2^o \neq 0$ holds for all integers k_1 and k_2 satisfying the inequality $|k_1| + |k_3| \leq 4$;
3: $D^o = -(\beta_{11}^o\alpha_2^{o2} - 2\beta_{12}^o\alpha_1^o\alpha_2^o + \beta_{22}^o\alpha_1^{o2}) \neq 0$, where $\beta_{\nu\mu}^o$ are the coefficients of the fourth form of the Hamiltonian $\bar{H}^o$, written in the following manner:

$$\bar{H}^o = \sum_{\nu=1}^{2} \frac{\alpha_\nu^o}{2} R_\nu + \sum_{\nu,\mu=1}^{2} \frac{\beta_{\nu\mu}^o}{4} R_\nu R_\mu + O_5, \quad R_\nu = \xi_\nu^2 + \eta_\nu^2 \tag{41}$$

where O_5 is a power series with minimum fifth order terms. Then the steady motion is Lyapunov stable.

This theorem will be applied using the normal form of the Hamiltonian

$$\bar{H}^o = H_2^{\text{new}} + H_4^{\text{new}} \tag{42}$$

where, H_2^{new} and H_4^{new} are given respectively by Eqs. 39 and 40.

7 Numerical simulations

Two types of satellites have been considered: M (medium) and S (small) types, whose orbital characteristics are similar to the American satellite PEGASUS (Crenshaw and Fitzpatrick 1968) and to the Brazilian data collecting satellites SCD1 and SCD2 (Kuga et al. 1999):

(a) Type M:

m (mass): 11,550 kg
a (orbital semi-major axis): 6959.64 km
e (orbital eccentricity): 0.0167
i (orbital inclination): 31.70°

(b) Type S:

m (mass): 100 kg
a (orbital semi-major axis): 7133.67 km
e (orbital eccentricity): 0.00178
i (orbital inclination): 24.99°

Several sets of parameters (combinations of values for the moments of inertia A, B, and C) were considered: one combination for the satellites of type M and six combinations for the satellites of type S.

For the simulations performed with the M type satellites, three equilibrium points were found (Table 1) and just one is found to be Lyapunov stable and it is described in Table 2. In this table L_2 is the modulus of the rotational angular momentum; J_2 is the angle formed by $\vec{L}_2$ and he component of $\vec{L}_2$in the Z axis of the satellite; **W** is the rotational speed of the satellite and **T** is the period of the rotation of the satellite.

For the simulations performed with the S type satellites 14 equilibrium points (Tables 3, 4, 5, 6, 7, 8), were found with only one stable solutions. The stable solutions found are given in Table 9. Table 10 gives a quantitative summary of the simulations performed.

Table 1 Satellite M type: $A = B = 3,9499 \times 10^{-1}$ kg km^2, $C = 1,0307 \times 10^{-1}$ kg km^2

Equilibrium points	Stable	Unstable	Unstable
L_1 (kg km^2/s)	-5.16995×10^{-2}	7.52494×10^{-7}	2.22282×10^{-5}
L_2 (kg km^2/s)	7.51908×10^{-4}	1.09457×10^{-6}	5.88142×10^{-5}
L_3 (kg km^2/s)	5.40662×10^{-4}	-8.84834×10^{-7}	1.91982×10^{-4}
l_2 (°)	45.01	−0.136	11.79
l_3 (°)	−31.74	28.25	271.83

Table 2 Lyapunov stable solution for M type satellites: $A = B > C$

A/B	A/C	A (kg km^2)	B (kg km^2)	C (kg km^2)
1	3.832249	3.9499×10^{-1}	3.9499×10^{-1}	1.0307×10^{-1}
L_2 (kg km^2/s)	J_2(°)	W (rpm)	T (s)	Stability
7.519081×10^{-4}	−39.4	0.07003316	86.128608	Stable

Table 3 Satellite S type: $A = B = 1.067 \times 10^{-5}$ kg km^2, $C = 1.3 \times 10^{-5}$ kg km^2

Equilibrium points	Unstable	Unstable	Unstable
L_1 (kg km^2/s)	-1.08994×10^{-5}	6.4457×10^{-6}	-2.25111×10^{-6}
L_2 (kg km^2/s)	9.82443×10^{-5}	4.7773×10^{-5}	1.64074×10^{-5}
L_3 (kg km^2/s)	-8.32162×10^{-5}	1.43643×10^{-5}	-4.90488×10^{-6}
l_2 (°)	1.055	−0.269	0.772
l_3 (°)	3671	4384.4	4204.9

Table 4 Satellite S type: $A = 1.067 \times 10^{-5}$ kg km^2, $B = 9.855 \times 10^{-6}$ kg km^2, $C = 1.3 \times 10^{-5}$ kg km^2

Equilibrium points	Stable
L_1 (kg km^2/s)	3.82102×10^{-6}
L_2 (kg km^2/s)	4.45568×10^{-6}
L_3 (kg km^2/s)	4.25629×10^{-6}
l_2 (°)	−49.44
l_3 (°)	209.6

8 Conclusions

In this work the Lyapunov stability of rotational motion of artificial satellites under the effects of the gradient torque is analyzed for nearly symmetrical satellites in near circular orbits. The canonical Andoyer variables are used to describe the rotational motion and they are shown to be appropriate for the application of the theorem of Kovalev and Savchenko. To apply this theorem, we first search for each satellite the equilibrium points numerically, their eigenvalues and select the stables ones. Secondly, the Hamiltonian F is expanded in Taylor series around each linear stable equilibrium. Subsequently, we apply a linear canonical transformation of variables followed by a complexification of these variables. Finally, the Lie–Hori

Table 5 Satellite S type: $A = B = 1.233 \times 10^{-5}\,\mathrm{kg\,km^2}$, $C = 1.450 \times 10^{-5}\,\mathrm{kg\,km^2}$

Equilibrium points	Unstable	Unstable	Unstable
L_1 $(\mathrm{kg\,km^2/s})$	2.11108×10^{-4}	-3.97028×10^{-5}	6.47819×10^{-6}
L_2 $(\mathrm{kg\,km^2/s})$	2.13833×10^{-4}	1.40829×10^{-4}	4.04328×10^{-5}
L_3 $(\mathrm{kg\,km^2/s})$	2.10534×10^{-4}	-1.07817×10^{-4}	9.07626×10^{-6}
l_2 (°)	−0.958	0.301	0.108
l_3(°)	−2234.4	2260.4	2253.4

Table 6 Satellite S type: $A = 1.233 \times 10^{-5}\,\mathrm{kg\,km^2}$, $B = 1.206 \times 10^{-5}\,\mathrm{kg\,km^2}$, $C = 1.05 \times 10^{-5}\,\mathrm{kg\,km^2}$

Equilibrium points	Unstable	Unstable	Unstable
L_1 $(\mathrm{kg\,km^2/s})$	7.14211×10^{-7}	8.14211×10^{-7}	-6.69303×10^{-7}
L_2 $(\mathrm{kg\,km^2/s})$	5.08332×10^{-6}	5.08332×10^{-6}	9.78378×10^{-7}
L_3 $(\mathrm{kg\,km^2/s})$	1.14306×10^{-6}	1.14306×10^{-6}	7.15488×10^{-7}
l_2 (°)	−1.464	−1.464	32.87
l_3(°)	−264.6	−264.6	9.276

Table 7 Satellite S type: $A = 1.233 \times 10^{-5}\,\mathrm{kg\,km^2}$, $B = 1.204 \times 10^{-5}\,\mathrm{kg\,km^2}$, $C = 1.05 \times 10^{-5}\,\mathrm{kg\,km^2}$

Equilibrium points	Unstable	Unstable
L_1 $(\mathrm{kg\,km^2/s})$	3.1441×10^{-7}	2.709×10^{-7}
L_2 $(\mathrm{kg\,km^2/s})$	1.7382×10^{-6}	5.16×10^{-7}
L_3 $(\mathrm{kg\,km^2/s})$	-6.587×10^{-7}	3.7601×10^{-7}
l_2 (°)	22.15	36.33
l_3 (°)	1.746	244.4

Table 8 Satellite S type: $A = B = 1.5 \times 10^{-6}\,\mathrm{kg\,km^2}$, $C = 1.24 \times 10^{-6}\,\mathrm{kg\,km^2}$

Equilibrium points	Unstable	Unstable
L_1 $(\mathrm{kg\,km^2/s})$	2.28744×10^{-6}	2.77579×10^{-7}
L_2 $(\mathrm{kg\,km^2/s})$	2.35365×10^{-6}	6.35354×10^{-7}
L_3 $(\mathrm{kg\,km^2/s})$	2.29794×10^{-6}	-4.23853×10^{-7}
l_2 (°)	187.3	−86.97
l_3 (°)	10.26	−13.71

algorithm is applied to obtain a normal form of order four and then the Lyapunov stability is examined.

Simulations are performed for two groups of artificial satellites, classified as medium and small satellites in agreement with their physical characteristics. It is considered that the satellites are in the perigee of their orbits. Equilibrium points are obtained for several initial conditions for each group of satellites. A normal form for the Hamiltonian is determined for each point and the three necessary conditions for Lyapunov stable motion are analyzed.

A restricted number of stable motions are observed in the simulations. This can be justified by the presence of several periodic terms considered in the Hamiltonian of the gravity

Table 9 Stable solutions for S type satellites: $B \approx A < C$

A/B	A/C	A (kg km^2)	B (kg km^2)	C (kg km^2)
1.08299	0.820769	10.67×10^{-6}	9.855×10^{-6}	13×10^{-6}
L_2 (kg km^2/s)	$J_2(°)$	W (rpm)	T (s)	Stability
4.455682×10^{-6}	49.14	3.290348	1.909581	Stable

Table 10 Quantitative summary of the simulations realized for each group of satellites associated to medium type satellite (M) and small type satellites (S)

Type of the satellite	Number of data Sets	Number of equilibrium points generated by the simulations	Number of stable motions	Number of unstable motions	
M	1	3	1	2	
				Failure in the 1st condition of the theorem	Failure in the 3rd condition of the theorem
				2	0
S	6	14	1	13	
				Failure in the 1st condition of the theorem	Failure in the 3rd condition of the theorem
				11	2

gradient torque or by the fact that the stable points found would be associated with the equilibrium points that are excluded due to singularities of the Andoyer variables. In fact, the small satellites possess orbital and physical characteristics similar to the Brazilian data collecting satellites that are stabilized by rotation corresponding to an inclination (Kuga et al. 1999; Orlando et al. 1997) angle $J_2 = 0$. Also, the considered medium type satellites have orbital and physics characteristics similar to the American satellite PEGASUS, which possesses a rotational tumbling motion (Crenshaw and Fitzpatrick 1968).

This work contributes for some mission's analysis of artificial satellites, pointing regions where the rotational motion is stable. This could be used to minimize attitude maneuvers to maintain a required attitude and thus leading to less fuel consumption, if the control uses a chemical propulsive system.

Acknowledgments The authors wish to express their appreciation for the support provided by CAPES, CNPq (under the contracts No. 308111/2006-0, 304841/2002-1, and 305147/2005-6).

Appendix

Disregarding the influence of the non-uniform mass distribution of the Earth, the potential of the translational-rotational motion of an artificial satellite under the influence of the gravity torque, truncated in the second order and degree of the associated Legendre polynomials, can be given by (Zanardi 1983; Kinoshita 1972):

$$U = -\frac{\mu M}{r} + \frac{\mu M}{r^3}\left[\frac{2C - A - B}{2M} P_2(\sin\beta) + \frac{A - B}{4M} P_2^2(\sin\beta)\cos 2\lambda\right] \tag{A.1}$$

where r is the distance of the center of mass of the satellite from the Earth's center, A, B, and C are the principal moment of the inertia of satellite, M is the mass of the satellite, β and λ are the latitude and the longitude of the mass center of the Earth referred to the system of principal axis of inertia of the satellite.

The development of the Legendre polynomial $P_2(\sin\beta)$ and $P_2^2(\sin\beta)$, and of $\cos 2\lambda$ in terms of the Andoyer and Delaunay variables (except for the mean anomaly) were presented by Hori (1971) and Kinoshita (1972). Cayley tables were used by Zanardi (1983) to express the true anomaly in terms of the mean anomaly.

The kinetic energy of the motion is given by translational kinetic energy (T_t) and rotational kinetic energy (T_r) (Hori 1971):

$$T = T_\mathrm{t} + T_\mathrm{r} \tag{A.2}$$

with

$$\begin{aligned} T_\mathrm{t} &= \frac{1}{2M}(\vec{p})^2 \\ T_\mathrm{r} &= \frac{1}{2}\left(A w_x^2 + B\, w_y^2 + C\, w_z^2\right) \end{aligned} \tag{A.3}$$

where $\vec{p}$ is the moment associated to the position vector $\vec{r}$, w_x, w_y, w_z are the components of the rotational angular velocity in the satellite's principal moment of inertia system. The components w_x, w_y, w_z are expressed in terms of the Euler angles ϕ, θ, ψ (which specify the relation between the satellite's principal moment of inertia system and the equatorial system) and their conjugate momenta p_ϕ, p_θ, p_ψ. It is possible to prove that the transformation $(\phi, \theta, \psi, p_\phi, p_\theta, p_\psi) \to (l_1, l_2, l_3, L_1, L_2, L_3)$ is canonical (Zanardi 1983; Hori 1971), with the momentum variables defined by:

$$L_1 = p_\psi = L_2 \cos J_2 \tag{A.4}$$

$$L_2 = \left(p_\theta^2 + p_\psi^2 + \left(\frac{p_\phi - p_\psi \cos\theta}{\sin\theta}\right)^2\right)^{1/2} \tag{A.5}$$

$$L_3 = p_\phi = L_2 \cos I I_2 \tag{A.6}$$

it means that the L_2 is the modulus of the rotational angular moment $\vec{L}_2$, L_1 is projection of $\vec{L}_2$ on the z-axis' principal axis system of inertia and L_3 is projection of $\vec{L}_2$ on the Z-equatorial axis, and the angular variables l_1, l_2, l_3 are angles, shown in the Fig. 1, which are related to the satellite system Oxyz (with axes parallel to the system of the spacecraft's principal axes of inertia) and the equatorial system OXYZ (with axes parallel to the axes of the Earth equatorial system).

Then the rotational kinetic energy can be expressed by:

$$\begin{aligned} T_\mathrm{r} = &\frac{1}{2}\left(\frac{1}{C} - \frac{1}{2A} - \frac{1}{2B}\right) L_1^2 + \frac{1}{4}\left(\frac{1}{A} + \frac{1}{B}\right) L_2^2 \\ &+ \frac{1}{4}\left(\frac{1}{B} - \frac{1}{A}\right)\left(L_2^2 - L_1^2\right)\cos 2l_1 \end{aligned} \tag{A.7}$$

It was considered also that through a canonical transformation of the variables of the orbital motion $(\vec{r}, \vec{p})$, where we get (Thiry 1970):

$$-\frac{\mu M}{r} + \frac{(\vec{p})^2}{2M} = -\frac{\mu^2 M^3}{2L^2} \tag{A.8}$$

The Hamiltonian of the translational–rotational motion can be taken the form:

$$F = F_0 + F_1, \tag{A.9}$$

where F_0 is the non disturbed Hamiltonian given by

$$F_0 = -\frac{\mu^2 M^3}{2L^2} + \frac{1}{2}\left(\left(\frac{1}{C} - \frac{1}{2A} - \frac{1}{2B}\right) L_1^2 + \frac{1}{2}\left(\frac{1}{A} + \frac{1}{B}\right) L_2^2\right) + \frac{1}{4}\left(\frac{1}{B} - \frac{1}{A}\right)\left(L_2^2 - L_1^2\right)\cos 2l_1 \tag{A.10}$$

and F_1 is the disturbance Hamiltonian due to the gravity gradient torque given by:

$$\begin{aligned}
F_1 = \frac{\mu^4 M^7}{L^6}\Bigg\{ & \frac{2C - A - B}{2M}\Bigg\{\{1 + 3e\cos l\}\left[P_2\left(\frac{L_1}{L_2}\right)\left[-\frac{1}{2} + \frac{3}{8}\left(1 + \theta^2 + \theta_2^2 - 3\theta^2\theta_2^2\right)\right.\right. \\
& -\frac{3}{8}\sin 2i\ \sin 2II_2\cos(h - l_3) - \frac{3}{8}\sin^2 i\ \sin^2\ II_2\cos(2h - 2l_3)\Big] \\
& -\frac{3}{16}(1 - 3\theta^2)\sin 2\ II_2\ \sin 2J_2\cos l_2 + \frac{3}{16}(1 - 3\theta^2)\ \sin^2\ II_2\ \sin^2 J_2\cos 2l_2 \\
& +\sum_\epsilon \frac{3}{16}\sin 2i(1 - \epsilon\theta_2)(1 + 2\epsilon\theta_2)\sin 2J_2\ \cos(h - l_3 - \epsilon l_2) \\
& +\sum_\epsilon \epsilon\frac{3}{16}\sin^2\ i\ \sin\ II_2(1 - \epsilon\theta_2)\sin 2J_2\ \cos(2h - 2l_3 + \epsilon l_2) \\
& -\sum_\epsilon \epsilon\frac{3}{16}\sin\ 2i\ \sin\ II_2(1 - \epsilon\theta_2)\sin^2\ J_2\ \cos(h - l_3 + 2\epsilon l_2) \\
& \left.-\sum_\epsilon \frac{3}{32}\sin^2 i(1 - \epsilon\theta_2)^2\sin^2\ J_2\ \cos(2h - 2l_3 + 2\epsilon l_2)\right] + P_2\left(\frac{L_1}{L_2}\right) \\
& \times\left\{\frac{3}{8}\sin^2 i(1 - 3\theta_2^2)\left[\cos(2l + 2g) + e\left[-\frac{1}{2}\cos(l + 2g) + \frac{7}{2}\cos(3l + 2g)\right]\right]\right. \\
& +\sum_\epsilon \epsilon\frac{3}{8}\sin\ i(1 + \epsilon\theta)\sin\ II_2 \\
& \times\left[\cos[2l + 2g + \epsilon(h - l_3)'] + e\left[-\frac{1}{2}\cos[l + 2g + \epsilon(h - l_3)]\right.\right. \\
& \left.\left.+\frac{7}{2}\cos\left[3l + 2g + \epsilon(2h - 2l_3)\right]\right]\right] \\
& -\sum_\epsilon \frac{3}{16}(1 + \epsilon\theta)^2\sin^2\ II_2\left[\cos[2l + 2g + \epsilon(2h - 2l_3)]\right. \\
& \left.\left.+e\left[-\frac{1}{2}\cos[l + 2g + \epsilon(2h - 2l_3)] + \frac{7}{2}\cos[3l + 2g + \epsilon(2h - 2l_3)]\right]\right]\right\}
\end{aligned}$$

$$+\sum_{\epsilon}\frac{9}{32}\sin^2 i\ \sin 2\,II_2\sin 2J_2\left\{\cos\left[2l+2g+\epsilon l_2\right]+e\left[-\frac{1}{2}\cos[l+2g+\epsilon l_3]\right.\right.$$

$$\left.\left.\left.+\frac{7}{2}\cos[3l+2g+\epsilon l_3]\right]\right]\right\}-\sum_{\epsilon\delta}\epsilon\frac{3}{16}\sin\ i(1+\epsilon\theta)\,(-\delta\theta_2)\,(1+2\delta\theta_2)$$

$$\times\sin\ 2J_2\left\{\cos[2l+2g+\epsilon(h-l_3)+\epsilon\delta l_2]+e\left[-\frac{1}{2}\cos[l+2g+\epsilon(h-l_3)+\epsilon\delta l_2]\right.\right.$$

$$\left.\left.+\ \frac{7}{2}\cos[3l+2g+\epsilon(h-l_3)+\epsilon\delta l_2]\right]\right\}+\sum_{\epsilon\delta}\delta\frac{3}{32}\sin\ II_2(1+\epsilon\theta)^2(1-\delta\theta_2)\sin\ 2J_2$$

$$\left\{\cos[2l+2g+\epsilon(2h-2l_3)+\epsilon\delta l_2]+e\left[-\frac{1}{2}\cos[l+2g+\epsilon(2h-2l_3)+\epsilon\delta l_2]\right.\right.$$

$$\left.\left.+\ \frac{7}{2}\cos[3l+2g+\epsilon(2h-2l_3)+\epsilon\delta l_2]\right]\right\}-\sum_{\epsilon}\frac{9}{32}\sin^2 i\ \sin^2 II_2\sin^2 J_2$$

$$\times\left\{\cos[2l+2g+2\epsilon l_2]+e\left[-\frac{1}{2}\cos[l+2g+2\epsilon l_2]\ +\frac{7}{2}\cos[3l+2g+2\epsilon l_2]\right]\right\}$$

$$+\sum_{\epsilon\delta}\epsilon\delta\frac{3}{16}\sin\ i(1+\epsilon\theta)\sin\ II_2(1-\delta\theta_2)\sin^2\ J_2\left\{\cos[2l+2g+\epsilon(h-l_3)+2\epsilon\delta l_2]\right.$$

$$\left.+e\left[-\frac{1}{2}\cos[l+2g+\epsilon(h-l_3)+2\epsilon\delta l_2]\ +\frac{7}{2}\cos[3l+2g+\epsilon(h-l_3)+2\epsilon\delta l_2]\right]\right\}$$

$$-\sum_{\epsilon\delta}\frac{3}{64}(1-\epsilon\theta)^2(1-\delta\theta_2)^2\sin^2\ J_2\left\{\cos[2l+2g+\epsilon(2h-2l_3)+2\epsilon\delta l_2]\right.$$

$$+e\left[-\frac{1}{2}\cos[l+2g+\epsilon(2h-2l_3)+2\epsilon\delta\, l_2]\right.$$

$$\left.\left.+\ \frac{7}{2}\cos[3l+2g+\epsilon(2h-2l_3)+2\epsilon\delta l_2]\right]\right\}\Bigg\}\Bigg\} \tag{A.11}$$

where:

$$\theta = \quad \cos i; \qquad i \text{ is the orbital plane inclination;} \tag{A.12}$$

$$\theta_2 = \frac{L_3}{L_2} = \cos\ II_2; \tag{A.13}$$

$$\frac{L_1}{L_2} = \cos\ J_2 \tag{A.14}$$

$P_2\left(\frac{L_1}{L_2}\right)$ are the Legendre polynomial of order 2;
$\sum\limits_{\epsilon}$ and $\sum\limits_{\epsilon,\delta}$ means that δ and ϵ assume values −1 and +1.

In the applications of this paper, the satellite is symmetrical, it means that $A = B$, or the satellite has the principal moment of inertia A close to the principal moment of inertia B.

References

Celletti, A., Sidorenko, V.: Some properties of the dumbbell satellite attitude dynamics. Celes. Mech. Dyn. Astron. **101**(1–2), 105–126 (2008)

Chudnenko, A.N.: On the stability of uniform rotations of a rigid body around the principal axis. Prikl. Mat. Mekh. **44**, 174–179 (1981)
Crenshaw, J.U., Fitzpatrick, P.M.: Gravity effects on the rotational motion of a uniaxial artificial satellite. AIAA J. **6**, 2140 (1968)
de Moraes, R.V.: A semi-analytical method to study perturbed rotational motion. Celes. Mech. **45**, 281–284 (1989)
Elipe, A., Ferrer, S.: On the equilibrium solutions in the circular planar restricted three rigid bodies problem. Celes. Mech. **37**(1), 59–70 (1985)
Elipe, A., Lopez-Moratalla, T.: On the Liapunov stability of stationary points around a central body. J. Guidance Control Dynam. **29**, 1376–1383 (2006)
Ferraz-Mello, S.: Canonical Perturbation Theories—Degenerate Systems and Resonance. Springer, New York (2007)
Hori, G.: Theory of general perturbations with unspecified canonical variables. Publ. Astron. Soc. Jpn. **18**, 287–299 (1966)
Hori, G.: Finite Two Body Problem, Reprinted from the Extra Collection of Papers Contributed to the IAU Symposium no. 48 (1971)
Kinoshita, H.: First order perturbations of the two finite body problem. Publ. Astron. Soc. Jpn. **24**, 423–439 (1972)
Kovalev, A.M., Savchenko, A.Ia.: Stability of uniform rotations of a rigid body about a principal axis. Prikl. Mat. Mekh. **39**(4), 650–660 (1975)
Kuga, H.K., Orlando, V., Lopes, R.V.F.: Flight dynamics operations during leap for the INPE's second environmental data collecting satellite SCD2. RBCM J. Brazilian Soc. Mech. Sci. **21**, 339–344 (1999)
Machuy, A.L.F.: Cálculo efetivo da forma normal parcial para o problema de Hill. 2001. M.Sc. Dissertation, 71p. Institute of Mathematics, Federal University of Rio de Janeiro, Rio de Janeiro (2001)
Mansilla, J.E.: Stability of Hamiltonian systems with three degrees of freedom and the three body problem. Celes. Mech. Dyn. Astron. **94**(3), 249–269 (2006)
Orlando, V., Kuga, H.K., Lopes, R.V.F.: Reducing the geopotential tesseral harmonic effects on autonomous longitude drift control of Sun-synchronous satellites. Adv. Astron. Sci. **95**, 361–374 (1997)
Sarychev, V.A., Mirer, S.A., Degtyarev, A.A., Duarte, E.K.: Investigation of equilibria of a satellite subjected to gravitational and aerodynamic torques. Celes. Mech. Dyn. Astron. **97**(4), 267–287 (2007)
Sarychev, V.A., Mirer, S.A., Degtyarev, A.A.: Equilibria of a satellite subjected to gravitational and aerodynamic torques with pressure center in a principal plane of inertia. Celes. Mech. Dyn. Astron. **100**(4), 301–318 (2008)
Stuchi, T.J.: KAM tori in the center manifold of the 3-D Hill problem. In: Winter, O.C., Prado, A.F.B. (eds.) Advanced in Space Dynamics, vol. 2, pp. 112–127. INPE, São José dos Campos (2002)
Thiry, U.: Les Fondements de La Mécanique Céleste. Gordon & Breach, Paris (1970)
Zanardi, M.C.F.P.S.: Coupled Translational-Rotational Motion of Artificial Satellites (in Portuguese). M.Sc. Dissertation, Aeronautical Institute of Technology, São José dos Campos (1983)
Zanardi, M.C.: Study of the terms of coupling between rotational and translational motion. Celes. Mech. **39**(2), 147–164 (1986)