Design of Non-Linear Controller for a flexible rotatory beam using State-Dependent Riccati Equation (SDRE) control

Pierre Bigot
National Institute for
Space Research- INPE
Sao jose dos campos, Brazil
Email: pierre.bigot.01@gmail.com

Luiz C. G. de Souza
National Institute for
Space Research- INPE
Sao jose dos campos, Brazil
Email: gadelha@dem.inpe

***Abstract*—Control of flexible structures is an open problem. Such structures can be very different, for example, robot arm or satellite solar panel. The common point between these structures is their very light weight and large length. Light structure control requires less energy but a much more complex control system to deal with vibrations. In this paper a flexible rotatory beam is modeled by Euler-Bernoulli hypothesis and its angular position is controlled. This kind of model is, most of the time, highly non-linear. As a result, controller designed by linear control technique can have its performance and robustness degraded. To deal with this problem, the State-Dependent Riccati Equation (SDRE) method is used to design and test a position control algorithm for the rigid-flexible non-linear model. The matlab/simulink simulator model is based on the characteristics of a real equipment. The control strategy uses a simple brushed DC electric motor. This work serves to validate the numerical simulator model and to verify the functionality of the control algorithm designed. In future work, this controller will be tested with the real rotatory beam to validate the model and the control algorithm.**

I. INTRODUCTION

Even if the design of flexible Euler-Bernoulli is a well-known problem, it is still a subject of research (M. Saad and Saydy, 2012). Moreover, most of the time, equations are linearized. Here, a first order non-linear kinematics is developped. The SDRE method (Souza and Gonzales, 2012) is an approach that can deal with non-linear plant; it linearizes the plant around the instantaneous point of operation and produces a constant state-space model of the system similar to LQR (Souza, 2008) control technique. The process is repeated in the next sampling periods therefore producing and controlling several state dependent linear models out of a non-linear one. For simplification, this work does not incorporate the Kalman filter technique; since it is assumed that all the states are known. Several simulations have proven the computationally feasibility for real time implementation (P.K. Menon and Cheng, 2002).

II. SDRE METHODOLOGY

Linear Quadratic Regulation (LQR) approach is well-known and its theory has been extended for the synthesis of non-linear control laws for non-linear systems (Souza, 2008). This is the case for satellite dynamics that are inherently non-linear. Several methodologies exist for control design and synthesis of these highly non-linear systems; these techniques include a large number of linear design methodologies (Isidori, 1995) such as Jacobian linearization and feedback linearization used in conjunction with gain scheduling (Shamma and Athens, 1990). Non-linear design techniques have also been proposed including dynamic inversion and sliding mode control (Slotine, 1996), recursive back stepping and adaptive control (K. Zhou and Glover, 1996).

Comparing with Multi-objective Optimization Non-linear control methods (I. Mainenti-Lopes and Sousa, 2012) the SDRE method has the advantage of avoiding intensive interaction calculation, resulting in simpler control algorithms more appropriated to be implemented in a satellite on-board computer.

The Non-linear Regulator problem (J. R. Cloutier and Mracek, 1996) for a system represented by the SDRE form with infinite horizon, can be formulated minimizing the cost function given by

$$J(x_0, u) = \frac{1}{2}\int_{t_0}^{\infty} (x^T Q(x)x + u^T R(x)u)\mathrm{d}t \qquad (1)$$

with the state $x \in R^n$ and control $u \in R^m$ subject to the non-linear system constraints given by

$$\begin{aligned} \dot{x} &= f(x) + B(x)u \\ y &= C(x)x + D(x)u \\ x(0) &= x_0 \end{aligned} \qquad (2)$$

where $B \in R^{n*m}$ and $C \in R^{s*n}$ are the system input and the output matrices respectively, and $y \in R^s$ where s is the dimension of the output vector of the system. D is the feedforward matrix and will be considered nul as in most of the sistems there is no direct action of the control on the output. $x(0)$ represents the initial conditions vector and $Q \in R^{n*n}$ and $R \in R^{m*m}$ are the weight matrices semi defined positive and defined positive respectively.

Applying a direct parameterization to transform the non-linear system into State Dependent Coefficients (SDC) representation (Souza and Gonzales, 2012), the dynamic equations of the system with control can be writen in the form

$$\dot{x} = A(x)x + B(x)u \tag{3}$$

with $f(x) = A(x)x$, where $A \in R^{n*n}$ is the states matrix. By and large $A(x)$ is not unique. In fact there is an infinite number of parameterizations for SDC representation. There are at least two parameterizations for all $0 \leq \alpha \leq 1$ satisfying

$$\alpha A_1(x)x + (1-\alpha)A_2(x)x + (1-\alpha)f(x) = f(x) \tag{4}$$

The choice of parameterizations must be made in accordance with the control system of interest. However, this choice should not violate the controllability of the system, i.e., the matrix controllability state dependent $[B(x) + A(x)B(x)A^{n-1}(x)B(x)]$ must be full rank.

The State-Dependent Algebraic Riccati Equation (SDARE) can be obtained applying the conditions for optimality of the variational calculus. In order to simplify expressions of big equations functions, $A(x)$, $B(x)$, $R(x)$, $Q(x)$ and $P(x)$ can be written without the reference to states x. As a result, the Hamiltonian for the optimal control problem given by Eq.(1) and Eq.(2) is

$$H(x,u,\lambda) = \frac{1}{2}(x^TQx + u^TRu) + \lambda^T(Ax + Bu) \tag{5}$$

where $\lambda \in R^n$ is the Lagrange multiplier. Applying to the Eq.(5) the necessary conditions for the optimal control given by $\dot{\lambda} = -\frac{\partial H}{\partial x}$, $\dot{x} = \frac{\partial H}{\partial \lambda}$ and $0 = \frac{\partial H}{\partial u}$ leads to

$$\begin{aligned} \dot{\lambda} = -Qx - \frac{1}{2}x^T\frac{\partial Q}{\partial x}x - \frac{1}{2}u^T\frac{\partial R}{\partial x}u \\ - \left[\frac{\partial Ax}{\partial x}\right]^T\lambda - \left[\frac{\partial Bu}{\partial x}\right]^T\lambda \end{aligned} \tag{6}$$

$$\dot{x} = A(x)x + B(x)u \tag{7}$$

$$0 = R(x)u + B(x)\lambda \tag{8}$$

Assuming the co-state in the form $\lambda = P(x)x$, which is dependent of the state, and using Eq.(8), the feedback control law is obtained as

$$u(x) = -R^{-1}(x)B^T(x)P(x)x \tag{9}$$

Substituting this result into Eq.(7) gives

$$\dot{x} = A(x)x - B(x)R^{-1}(x)B^T(x)P(x)x \tag{10}$$

To find the function P, $\lambda = P(x)x$ is differentiated with respect the time along the path

$$\dot{\lambda} = \dot{P}x + PAx - PBR^{-1}B^TPx \tag{11}$$

Substituting Eq.(11) in the first necessary condition of optimal control Eq.(6) and arranging the terms more appropriately results in

$$\begin{aligned} 0 = \dot{P}x + \frac{1}{2}x^T\frac{\partial Q}{\partial x} + \frac{1}{2}u^T\frac{\partial R}{\partial x}u \\ + x^T\left[\frac{\partial A}{\partial x}\right]^T Px + \left[\frac{\partial Bu}{\partial x}\right]^T P \\ + (PA + A^TP - PBR^{-1}BP + Q)x \end{aligned} \tag{12}$$

Two important relations are obtained to satisfy the equality of Eq.(12). The first one is state-dependent algebraic Riccati equation (SDARE) which solution is $P(x)$ given by

$$PA + A^TP - PBR^{-1}B^TP + Q = 0 \tag{13}$$

The second one is the necessary condition of optimality which must be satisfied, given by

$$\begin{aligned} 0 = \dot{P}x + \frac{1}{2}x^T\frac{\partial Q}{\partial x} + \frac{1}{2}u^T\frac{\partial R}{\partial x}u \\ + x^T\left[\frac{\partial A}{\partial x}\right]^T Px + \left[\frac{\partial Bu}{\partial x}\right]^T P \end{aligned} \tag{14}$$

Finally, the non-linear feedback control is by

$$u = -R^{-1}(x)B^T(x)P(x)x \tag{15}$$

For some special cases, such as systems with little dependence on the state or with few state variables, Eq.(13) can be solved analytically. On the other hand, for more complex systems, the numerical solution can be obtained using an adequate sampling rate. It is assumed that the parameterization of the coefficients dependent on the state is chosen so that the pairs $(A(x), B(x))$ and $(C(x), A(x))$ are in the linear sense for every x belonging to the neighborhood about the origin, point to point, stabilizable and detectable, respectively. Then the SDRE non-linear regulator produces a closed loop solution that is locally asymptotically stable. An important factor of the SDRE method is that it does not cancel the benefits that result from the non-linearities of the dynamic system, because, it is not require inversion and no dynamic feedback linearization of the non-linear system.

III. ROTATORY BEAM'S MODEL

A. Beam definition

Figure (1) shows a representation of a flexible rotatory beam; it consists of a beam fixed to the rotor motor at one end and free at the other one. Euler-Bernoulli beam is used, this means that deformations are considered small. Parameters of the beam are the following: length L, linear density ρ, rigidity EI_z and the rotor motor parameters are: angular position $\theta(t)$, which is a rotation along the x-axis so gravity has no influence, inertia J_m , torque Γ_m and radius r. The beam displacement is $y(x,t)$ and the deflection angle is $\alpha(x,t)$. To simplify notation, y and α are used without referring to their variables and their

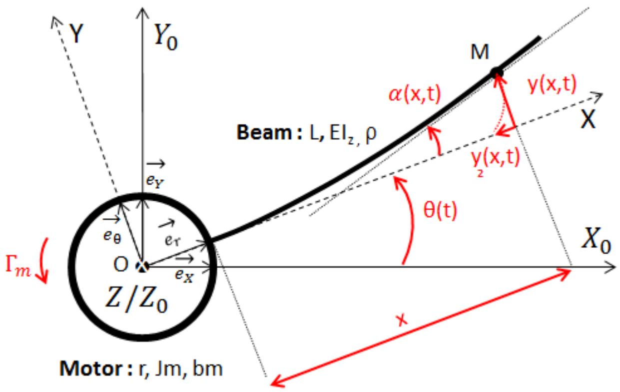

Fig. 1: Representation of the flexible rotatory beam

partial derivative relative to the time t and the abscise x are respectively written $\dot{y}$ and y'.

It can be noted than the deflection angle is related with de displacement according to

$$\alpha(x,t) = \frac{\partial y(x,t)}{\partial x} = y' \tag{16}$$

B. Kinematics

Let M be a point of the beam. In the inertial reference system $R(X,Y,Z)$, coordinates of M are

$$\overrightarrow{OM} = \begin{bmatrix} r+x \\ y \end{bmatrix}_R \tag{17}$$

The velocity of this point is the derivative with respect to the fix reference system $R_0(X_0,Y_0,Z_0)$. The beam is considered inextensible, so $\dot{x} = 0$.

$$\begin{aligned} \overrightarrow{v_m} &= \left.\frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t}\right]_{R_0} = \left.\frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t}\right]_R + \vec{\Omega}_{R/R_0} \otimes \overrightarrow{OM}_R \\ &= \begin{bmatrix} -y\dot{\theta} \\ (r+x)\dot{\theta} + \dot{y} \end{bmatrix}_R \end{aligned} \tag{18}$$

C. Kinetic and potential energies

Kinetic energy of this system can be represented by two terms. The first one due to the motor rotation and the other due to the beam rigid-flexible motion.

$$\begin{aligned} T_{rotor} &= \frac{1}{2} J_m \dot{\theta}^2 \\ T_{beam} &= \frac{1}{2}\rho \int_0^L \overrightarrow{v_m}.\overrightarrow{v_m} \mathrm{d}x \end{aligned} \tag{19}$$

Second or more order in displacement y such as the axial displacement $y_2(x,t)$ wont be considered because of small deformations hypothesis. After doing some calculation, Eq.(19) becomes.

$$\begin{aligned} T = &\frac{1}{2}\dot{\theta}^2 \left(J_m + \frac{1}{3}\rho(r+L)^3 + \rho\int_0^L y^2 \mathrm{d}x \right) \\ &+ \dot{\theta}\rho \int_0^L (r+x)\dot{y}\mathrm{d}x + \frac{1}{2}\int_0^L \dot{y}^2 \mathrm{d}x \end{aligned} \tag{20}$$

The potential energy of a flexible beam is given by

$$V = \frac{1}{2} EI_z \int_0^L y''^2 \mathrm{d}x \tag{21}$$

In order to use energies to write the equations of motion, the beam deformation variable, that is, the displacement y, is need to be known explicitely. To do that, the assumed modes method is used.

D. Assumed modes

The motor rotation produces beam transverse vibrations. Analysing an infinitesimal element of the beam and considering moments and forces acting on it, as shown in figure(2) is obtained. Variable Q is the shear force, variable M and ρ is the linear density.

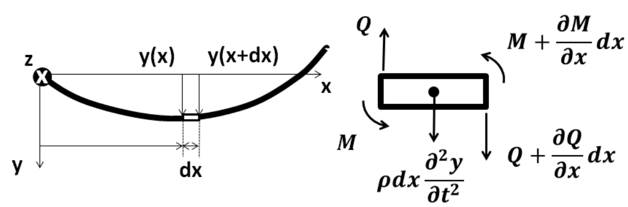

Fig. 2: Forces affecting the flexible beam

The application of the fundamental principle of the dynamics leads to Eq.(22), where the first one is the force in the direction of the axis y and the second one is the moment along the axis z.

$$\frac{\partial Q}{\partial x} = \rho \mathrm{d}x \frac{\partial^2 y}{\partial t^2} \qquad Q = \frac{\partial M}{\partial x} \tag{22}$$

Moreover, for a prismatical beam, the rigidity (EI_z) is a constant so

$$M = -EI_z \frac{\partial^2 y}{\partial x^2} \tag{23}$$

Combining Eq.(22) and Eq.(23) leads to the general equation for transverse vibration of an uniform beam.

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho}{EI_z}\frac{\partial^2 y}{\partial t^2} = 0 \tag{24}$$

Looking for a solution for this equation as a product of temporal and spatial function of the form $y(x,t) = \Phi(x)q(t)$ given by

$$\begin{aligned} \Phi(x) &= A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x \\ q(t) &= E\cos\omega t + F\sin\omega t \\ \beta^4 &= \frac{\rho\omega^2}{EI_z} \end{aligned} \tag{25}$$

Boundary conditions at beam ends are essential to determine the shape function Φ and parameters A, B, C and D. As the beam is clamped to the rotor, displacement and deflection are null at $x = 0$ (Eq.(26)). Likewise, the shear force and bending moment are zero at $x = L$ (Eq.(27)).

$$\Phi(0) = 0 \qquad \left.\frac{\partial \Phi}{\partial x}\right|_{x=0} = 0 \tag{26}$$

$$\left.\frac{\partial^3 \Phi}{\partial x^3}\right|_{x=L} = 0 \qquad \left.\frac{\partial^2 \Phi}{\partial x^2}\right|_{x=L} = 0 \tag{27}$$

Substituting Φ from Eq.(25) into Eq.(26) and Eq.(27) gives a system of four equations. This system can be easily reduced to one equation called characteristic equation.

$$\cos\beta L \cosh\beta L = -1 \tag{28}$$

The solution of Eq.(28) is an infinite set of spatial natural pulsations β_i where i is the mode number. The shape function Φ from Eq.(25) associated to the mode i, called Φ_i, can now be written analytically. The first four mode shape are plotted in Figure(3).

$$\begin{aligned} \Phi_i(x) &= A_i \left[\cosh\beta x - \cos\beta x + k_i(\sinh\beta x - \sin\beta x)\right] \\ with \quad k_i &= \frac{\sin\beta L - \sinh\beta L}{\cos\beta L + \cosh\beta L} \end{aligned} \tag{29}$$

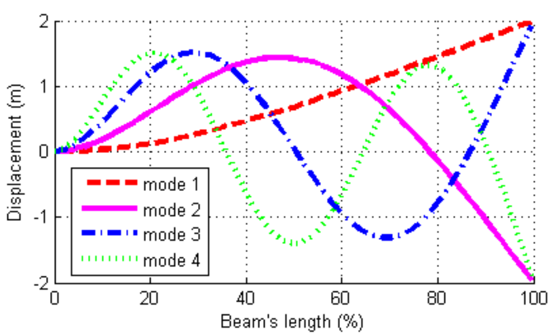

Fig. 3: First four shape function$\Phi(x)$

A finite number, of modes n is assumed to shape the beam deformation. The solution, Eq.(30), is a linear combination of all these modes.

$$y(x,t) = \sum_{i=1}^{n} \Phi_i(x) q_i(t) = \Phi^T q = q^T \Phi \tag{30}$$

Now that the displacement $y(x,t)$ is known explicitely, motion equations can be written using the Lagrange theory.

E. Dynamic equations

In this section, Lagrange theory is used to develop motion equations. The generalized coordinates are the rigid motion, θ, and the flexible modes, q, denoted by $p = \begin{bmatrix} p_1 & p_2 \end{bmatrix}^T = \begin{bmatrix} \theta & q \end{bmatrix}^T$. External force F along the axis y is considered null and along θ is equal to Γ_{rotor} , the rotor torque.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{p}_i} - \frac{\partial T}{\partial p_i} + \frac{\partial V}{\partial p_i} = F_{p_i} \text{ where } i \in [1,2] \tag{31}$$

Substituing the expression of the displacement of Eq.(30) in kinetic and potential energies (Eq.(20) and Eq.(21) respectively)

$$\begin{aligned} T = &\frac{1}{2}\dot{\theta}^2 \left(J_m + \frac{1}{3}\rho(r+L)^3 + \rho \int_0^L q^T \Phi \Phi^T q \mathrm{d}x \right) \\ &+ \dot{\theta}\rho \int_0^L (r+x)\dot{q}^T \Phi \mathrm{d}x + \frac{1}{2}\int_0^L \dot{q}^T \Phi \Phi^T q \mathrm{d}x \end{aligned} \tag{32}$$

$$V = \frac{1}{2} E I_z \int_0^L q^T \Phi'' \Phi''^T q \mathrm{d}x \tag{33}$$

and combining these results with Eq.(31) leads to the two Lagrange equations, according to the generalized coordinates θ and q:

$$\begin{aligned} \Gamma_m = &\ddot{\theta}\left(J_m + \frac{1}{3}\rho(r+L)^3 + \rho \int_0^L q^T \Phi \Phi^T q \mathrm{d}x \right) \\ &+ 2\dot{\theta}\rho \int_0^L q^T \Phi \Phi^T q \mathrm{d}x + \rho \int_0^L (r+x)\ddot{q}^T \Phi \mathrm{d}x \\ 0 = &\ddot{\theta}\rho \int_0^L (r+x)\Phi \mathrm{d}x + \rho \int_0^L \Phi \Phi^T \ddot{q} \mathrm{d}x \\ &+ \dot{\theta}^2 \int_0^L \Phi \Phi^T q \mathrm{d}x + E I_z \int_0^L \Phi'' \Phi''^T q \mathrm{d}x \end{aligned} \tag{34}$$

that can be expressed on a matrix format

$$\begin{aligned} \begin{bmatrix} \Gamma_{rotor} \\ 0 \end{bmatrix} = &\begin{bmatrix} J_{eq} + q^T M_{ff} q & M_{rf}^T \\ M_{rf} & M_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{q} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_{ff} \end{bmatrix} \begin{bmatrix} \theta \\ q \end{bmatrix} \\ &+ \begin{bmatrix} q^T M_{ff} \dot{q} & q^T M_{ff} \dot{\theta} \\ -M_{ff} q \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{q} \end{bmatrix} \end{aligned} \tag{35}$$

with the following parameters

$$\begin{aligned} M_{ff} &= \rho \int_0^L \Phi \Phi^T \mathrm{d}x \\ M_{rf} &= \rho \int_0^L (r+x) \Phi \mathrm{d}x \\ K_{ff} &= E I_z \int_0^L \Phi'' \Phi''^T q \mathrm{d}x \\ J_{eq} &= J_m + \frac{1}{3}\rho(r+L)^3 \end{aligned} \tag{36}$$

Finally, Eq.(35) can be written on a more compact manner definning M, C_o K and F, as matrix of mass, coriolis, rigidity and external force respectively

$$M\ddot{p} + C_o \dot{p} + Kp = F \tag{37}$$

F. Motor DC

This system is controlled with the voltage delivered by the motor, thus, it is needed to express the rotor torque Γ_{rotor} in function of the motor supply voltage U_m.

A classical model for a DC motor, taking losses into acount, is giving in Eq.(38). The parameters involved in this equation are: the friction constant between the rotor and stator b_m, the efficiencie of the motor η_m, the efficiencie of gears η_g, the motor torque constant K_t, the transmission constant of gears K_g, the back EMF constant K_m and the motor armature resistence R_m. All these parameters leads to the well known motor equation

$$\Gamma_{rotor} = \frac{\eta_m \eta_g K_t K_g}{R_m}(U_m - K_g K_m \dot{\theta}) - b_m \dot{\theta} \tag{38}$$

Defining C_m as

$$C_m = \frac{\eta_m \eta_g K_t K_g}{R_m} \tag{39}$$

it is now possible to explicit the vector of external forces $F = \begin{bmatrix} \Gamma_{rotor} & 0 \end{bmatrix}^T$ as

$$\begin{aligned} F &= \begin{bmatrix} C_m \\ 0 \end{bmatrix} U_m - \begin{bmatrix} C_m K_g K_m + b_m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{q} \end{bmatrix} \\ &= L U_m - N\dot{p} \end{aligned} \tag{40}$$

where N and L are two matrix whose values can easily be identified in this same equation. Substituing the result from Eq.(40) in Eq.(37) gives

$$M(p)\ddot{p} + (C_o(p,\dot{p}) + N)\dot{p} + Kp = LU_m \tag{41}$$

Thus, in the global matrix equation of the system appears an additive damping term $N\dot{p}$.

According to small deformation hypotesis, the non-linear term $q^T M_{ff} q$ in the mass matrix M is really small, so it can be negligible. Whereas, no terms in the coriolis matrix C_o are negligible since $\dot{\theta}$ and $\dot{q}$ are not necessarily small (M. Saad and Saydy, 2012).

To check the validity of these assumptions Figure (4) represents the impulse response (1 second, amplitude 5V) to three different models: the linear model, the fully non-linear model and the non-linear model without non-linear term in the mass matrix M (partially non-linear). As it is possible to notify, fully and partially non-linear model have almost the same response whereas the linear model is quite different.

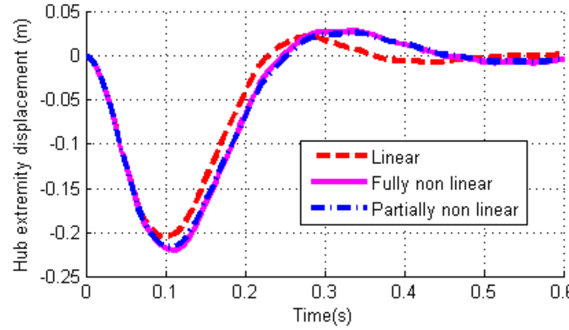

Fig. 4: Comparison between different plant models

IV. SIMULATION STRATEGY

A. State space model

For purpose of simulation and control, this system is represented using the state space model.

$$\begin{aligned} \dot{x} &= A(x)x + Bu \\ y &= Cx \end{aligned} \tag{42}$$

where states vector x and control u are defined by

$$x = \begin{bmatrix} \theta & q & \dot{\theta} & \dot{q} \end{bmatrix}^T = \begin{bmatrix} p & \dot{p} \end{bmatrix}^T \qquad u = U_m \tag{43}$$

So, reorganizing Eq.(41), the classic state space representation is obtained.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0_{n+1} & I_{n+1} \\ -M^{-1}K & -M^{-1}(N + N_{nolin}) \end{bmatrix}^T x + \begin{bmatrix} 0_{(n+1,1)} \\ -M^{-1}L \end{bmatrix}^T U_m \\ &= A(x)x + BU_m \end{aligned} \tag{44}$$

with n, the number of flexible modes, and I, the identity matrix. Studing the equation above, matrix A depends on the state and matrix B is a constant.

For the simulation it is considered that all states are available, then C will be the identity matrix.

This representation is adequate to use the SDRE theory explained in section II

B. SDRE implementation

The algorithm is desribed in the Figure (5). As the matrix A depends on the states it must be determined on every step. So, for every iteration of the simulation, states vector X is measured , the Riccati solution P is obtained from Eq.(11), the feedback control u is determined thanks to Eq.(15) and then, the new matrix A is obtained.

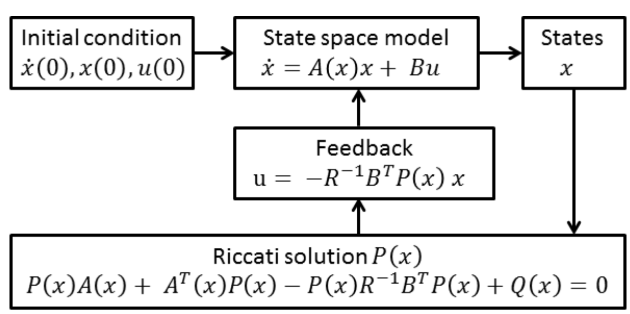

Fig. 5: SDRE algorithm

Implementation of this algorithm has been done using the MATLAB-Simulink. Riccati equation has been determinated via the LqrSim block (Campa, 2001). Figure(6) represents the Simulink solution for the feedback control u.

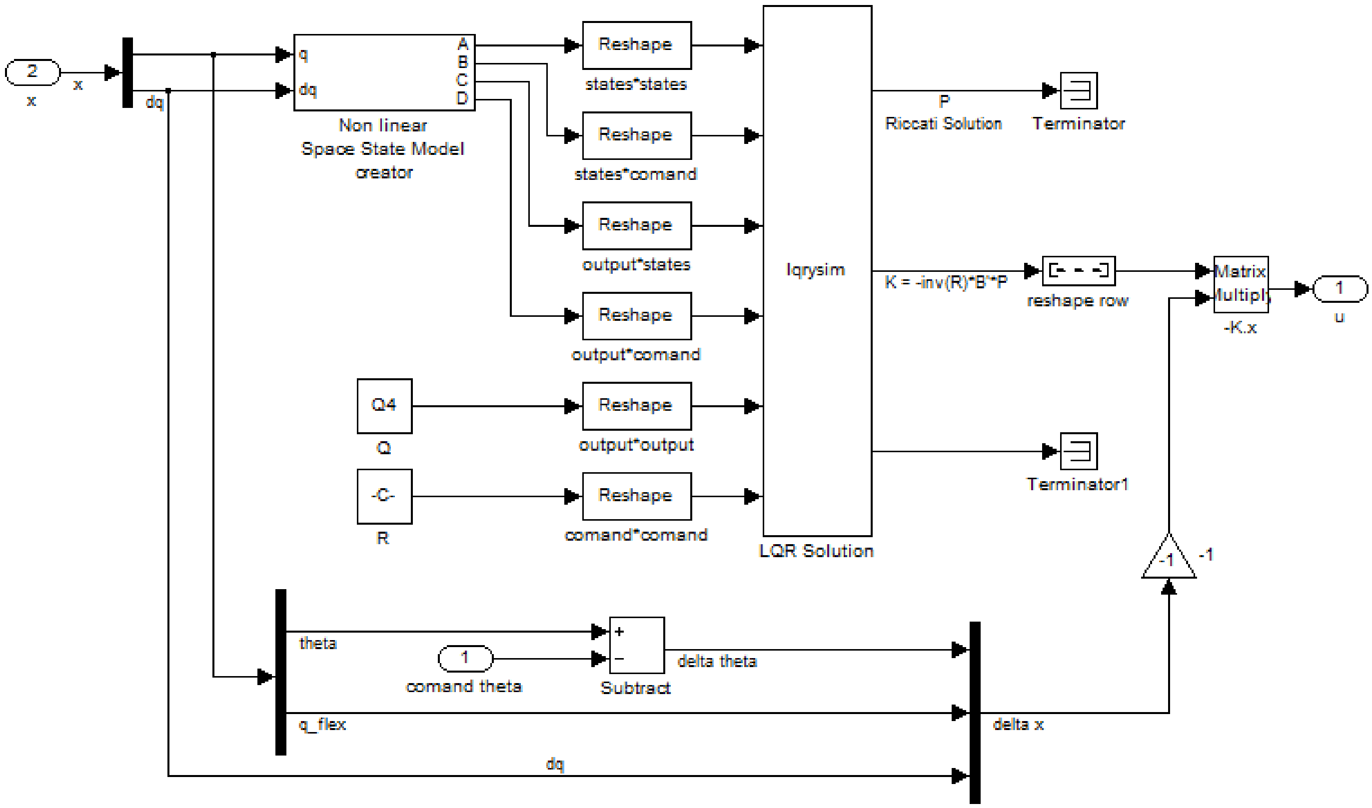

Fig. 6: Simulink feedback calculus with SDRE algorithm

C. *Performance requirements*

Due to the physical features of the system, maximum voltage supply of the motor is 5V. Refering to performance objectives, those are temporal requirements since the model is non-linear and frequency analysis is not possible. For the rigid motion, only one overshoot of the beam angular position is accepted, afterwhat it shall be stabilized in the region $\pm 2\%$ in a minimum setting time. To analyse the flexible motion the displacement at beam's extremity is measured. As small deformation hypothesis may not be infringed, maximum overshoot is $\pm 0.05m$ which corresponds to 10% of the beam length. Table (I) sumarizes requirements defined in this section.

Parameter	Performance	Condition
θ	overshoot	only one
θ	rise time	minimum
$y(x = L)$	overshoot	$\leq$ 0.05 m
U_m	voltage	$\leq$ 5 V

TABLE I: Performance requirements

V. SIMULATION RESULTS

A. *Model values*

Table (II) shows the values used for the simulation.

Motor parameters	values
L	0.4826 m
J_m	0.002 $\mathrm{kg\,m^2}$
ρ	0.1347 $\mathrm{kg\,m^{-1}}$
Beam parameters	values
b_m	0.004 $\mathrm{kg\,m^2\,s^{-1}}$
EI_z	0.54 $\mathrm{N\,m}$
C_m	0.1527 $\mathrm{N\,V^{-1}}$

TABLE II: Model parameter values

B. *Vibration mode analysis*

To know the vibration frequencies, an eigenvalue analysis of the full system represented by Eq.(41) is done. For this analysis damping and external forces are considered null. Results are shown in table (III).

Mode	1	2	3	4	5
f (Hz)	4.58	16.08	42.70	83.00	136.88

TABLE III: Vibration frequencies

In this simulation, only the rigid mode and the first three flexible modes are taking into acount.

C. *Variation of parameters Q and R*

SDRE technique is the generalized method of LQR for state dependent equation. In this paper, LQR parameters, R and Q, are constant since they do not depend on the states. The controller performance is directly related with these weights R and Q.

From the cost function represented by Eq.(1) it can be noted that matrix Q is linked with the states x and matrix R with the control signal U_m. In order to influence at each state separately Q matrix is chosen diagonal. Thus, each diagonal term is related to one state and acts as a penalty: the higher the value, less the influence of the state. The matrix R, (here a scalar because there is only one control variable) allows to size the control signal. In the same way as for Q, a high value of R penalises the control signal.

In order to choose these parameters a set of Q and R values are tested. It has been notified during experiments that modifications on R influences almost only the control signal and not the dynamic response. For these reason, to determine Q and R values, first a set of Q values are tested; then, using the value that best matches the dynamical requirements, a set of R values are experimented to find the one which fulfilled the motor requirement: motor voltage $U_m < 5$V.

From the results of this simulation it is possible to determine the influence of each term inside Q on the output. The first one related to θ reduces the setting time but increases overshoot in angle position and displacement. The second one, related to flexible states does not show a significative influence. The third one, related to the angular velocity helps to reduce slightly overshoot in θ. Finally, the last one related to the derivative of flexible states appears to reduce significatly displacement overshoot. Table (IV) represents the evolution of Q values from Q_1 to Q_4, which have been changed acording to the logic described above to finally get Q_4; the best Q fulfilling our requirements. which is fully explicit in Eq.(45). The choosen value for R is 5.

$$\begin{aligned} Q &= diag(100, 1, 1, 1, 1, 320, 320, 320) \\ R &= 5 \end{aligned} \tag{45}$$

Responses for a step input of 60 ° are shown in figure (7), and, figure (8) shows that the value of $R = 5$ enabled to fit the condition $U_m < 5V$.

Q	θ	q	$\dot{\theta}$	$\dot{q}$
$Q1$	1	1	1	1
$Q2$	100	1	1	1
$Q3$	100	1	5	320
$Q4$	100	1	1	320

TABLE IV: Parameters Q tested

VI. CONCLUSIONS

In this paper, the model of a rotatory Euler-Bernoulli beam is successfully built and the required performance objectives and physical requirements are achieved. This study shows how to implement a SDRE (State-Dependent Riccati Equation) controller for simulation. This controller model can be useful to simulate many other non-linear systems. The SDRE is tested here with a simple model for a flexible rotatory beam. One of the main interest of this work is that, changing values of physical parameters such as beam length, or inertia, this model can easily be extended to much more complex systems such as satellite solar panels or robotic arm. In a future work this controller will be tested with the real rotatory beam in order to verify the real time implementation feasibility.

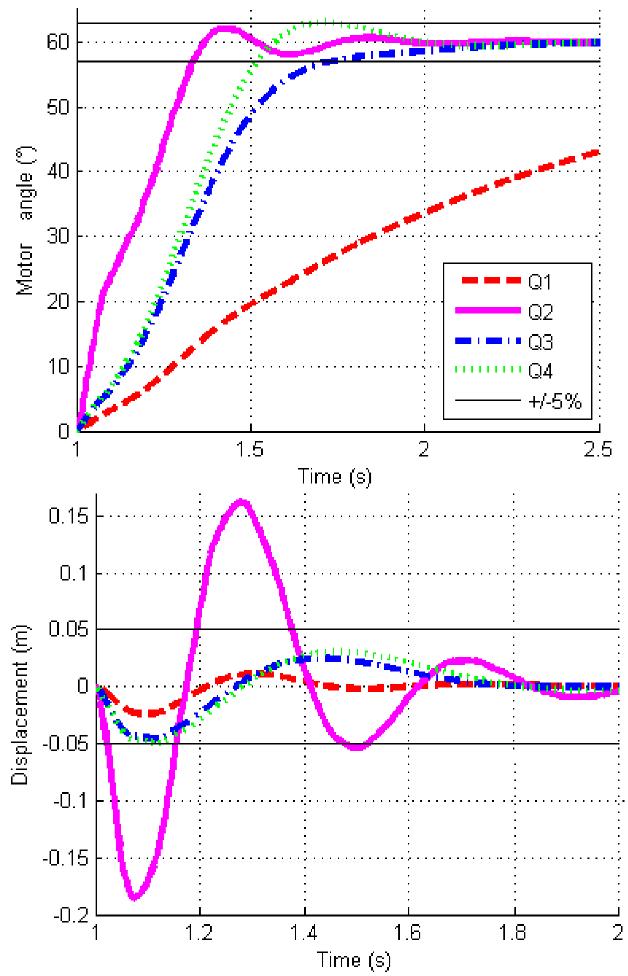

Fig. 7: Step response of the linear plant for differents Q

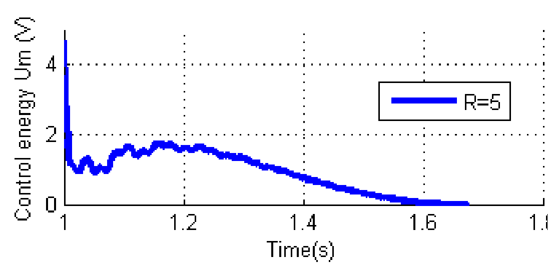

Fig. 8: Control energy U_m for $R = 5$

ACKNOWLEDGEMENTS

The Brasilian fundation for scientific and technological research CNPq for the scholarship master degree.

REFERENCES

Campa, G. (2001). Algebraic riccati equation solution in simulink via c+fortran. *Matlab Central File Exchange*.

I. Mainenti-Lopes, L. C. G. S. and Sousa, F. L. (2012). Design of a nonlinear controller for a rigid-flexible satellite using multi-objective generalized extremal optimization with real codification. *Shock and vibration*, 19:1–10.

Isidori, A. (1995). *Nonlinear Control Systems*. Springer-Verlag, New York, 1st edition.

J. R. Cloutier, C. D. S. and Mracek, C. P. N. (1996). Nonlinear regulation and nonlinear h-infinity control via the state-dependent riccati equation technique: Part 1. theory. In *Proceedings of the First International Conference on Nonlinear Problems in Aviation and Aerospace*.

K. Zhou, J. D. and Glover, K. (1996). *Robust and Optimal Control*. Prentice Hall, Englewood, New Jersey, 1st edition.

M. Saad, O. A. and Saydy, L. (2012). Analytical model of one flexible link system with nonlinear kinematics. *Journal of Vibration and Control*.

P.K. Menon, T. Lam, L. S. C. and Cheng, V. H. L. (2002). Real time computational methods for sdre non-linear control of missiles. In *American Control Conference*.

Shamma, J. S. and Athens, M. (1990). Analysis of gain scheduled control for nonlinear plants. In *IEEE Trans. on Auto. Control*.

Slotine, J.-J. E. (1996). *Applied nonlinear control*. Prentice Hall, Englewood, New Jersey, 1st edition.

Souza, L. C. G. (2008). Design of satellite control system using optimal non-linear theory. *Mechanics Based Design of Structures and Machine*.

Souza, L. C. G. and Gonzales, R. G. (2012). Application of the state-dependent riccati equation and kalman filter techniques to the design of a satellite control system. *Shock and vibration*.